

# STABLE GROTHENDIECK RINGS OF WREATH PRODUCT CATEGORIES

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**ABSTRACT.** Let  $k$  be an algebraically closed field of characteristic zero, and let  $\mathcal{C} = \mathcal{R} - \text{mod}$  be the category of finite-dimensional modules over a fixed Hopf algebra over  $k$ . One may form the wreath product categories  $\mathcal{W}_n(\mathcal{C}) = (\mathcal{R} \wr S_n) - \text{mod}$  whose Grothendieck groups inherit the structure of a ring. Fixing distinguished generating sets (called basic hooks) of the Grothendieck rings, the classification of the simple objects in  $\mathcal{W}_n(\mathcal{C})$  allows one to demonstrate stability of structure constants in the Grothendieck rings (appropriately understood), and hence define a limiting Grothendieck ring. This ring is the Grothendieck ring of the wreath product Deligne category  $\text{Rep}(\mathcal{R} \wr S_t)$ . We give a presentation of the ring and an expression for the distinguished basis arising from simple objects in the wreath product categories as polynomials in basic hooks. We discuss some applications when  $\mathcal{R}$  is the group algebra of a finite group, and some results about stable Kronecker coefficients. Finally, we explain how to generalise to the setting where  $\mathcal{C}$  is a tensor category.

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## 1. INTRODUCTION

In this paper, we consider the Grothendieck groups of the wreath product  $\mathcal{R} \wr S_n$ , where  $\mathcal{R}$  is a Hopf algebra, and  $S_n$  is the symmetric group on  $n$  symbols. The Hopf algebra structure gives rise to a multiplication, so we may speak of Grothendieck rings. Using a description of the simple modules for such wreath products

in terms of induced modules, we use Mackey theory to understand the multiplication on the Grothendieck ring in terms of data associated to  $\mathcal{R}$  and  $S_n$ . We show that this multiplication exhibits a certain stability property which allow us to define a “limiting Grothendieck ring”,  $\mathcal{G}_\infty(\mathcal{C})$  (here  $\mathcal{C} = \mathcal{R} - \text{mod}$ ). This ring is the Grothendieck ring of the wreath product Deligne categories  $S_t(\mathcal{C})$  considered in [Har16]. When  $\mathcal{C}$  is the category of finite-dimensional vector spaces over  $k$ , we recover the original Deligne category  $\text{Rep}(S_t)$ , and when  $\mathcal{C} = \mathcal{R} - \text{mod}$ , we write  $S_t(\mathcal{C}) = \text{Rep}(\mathcal{R} \wr S_t)$ . Deligne proved that the Grothendieck ring of  $\text{Rep}(S_t)$  is the free polynomial algebra on certain elements that we call basic hooks (see [Del07]). We define a collection of elements  $T_n(U)$  indexed by positive integers and simple  $\mathcal{R}$ -modules, such that for fixed  $n$  they span a Lie algebra isomorphic to the Grothendieck ring of  $\mathcal{R} - \text{mod}$ , which we write as  $\mathcal{G}(\mathcal{C})_n$ . We show that

$$\mathcal{G}_\infty(\mathcal{C}) = \bigotimes_{i=1}^{\infty} U(\mathcal{G}(\mathcal{C})_i)$$

Here,  $U(\mathcal{G}(\mathcal{C})_i)$  is the universal enveloping algebra of the span of  $T_i(U)$  where  $U$  are the simple  $\mathcal{R}$ -modules. If  $\mathcal{G}_\infty(\mathcal{C})$  is thought of as the Grothendieck ring of the wreath product Deligne category  $S_t(\mathcal{C})$ , there is a natural basis,  $X_\lambda$ , coming from the images of the indecomposable objects of  $S_t(\mathcal{C})$ . We give a generating function that relates this distinguished basis of  $\mathcal{G}_\infty(\mathcal{C})$  to the  $T_n(U)$ . The generating function is:

$$\sum_{\lambda \in \mathcal{P}_{\mathcal{C}}} \left( \prod_{U \in I(\mathcal{C})} s_{\lambda(U)}^{(U)} \right) \otimes X_\lambda = \left( \sum_{r \geq 0} (-1)^r e_r^{(1)} \right) \prod_{l=1}^{\infty} \exp \left( \sum_{U' \in I(\mathcal{C})} \left\langle [U'], \log \left( 1 + \sum_U p_l^{(U)} [U] \right) \right\rangle \otimes T_l(U') \right)$$

In some sense, this reduces the structure of  $\mathcal{G}_\infty(\mathcal{C})$  to data associated to  $\mathcal{C}$  and symmetric function combinatorics. Finally, we discuss implications of these results for the asymptotic representation theory of wreath products and symmetric groups, and explain that all our results actually hold when  $\mathcal{C}$  is a tensor category, without the need for a Hopf algebra  $\mathcal{R}$ .

The outline of the paper is as follows. In Section 3 we establish notation relating to symmetric functions, then in Section 4 we discuss wreath products and their irreducible representations. In Section 5 we apply Mackey theory to show tensor products of these simple objects decompose in ways controlled by double coset representatives of Young subgroups of symmetric groups which we discuss in Section 6. We see that these exhibit certain stability properties that allow us to define a “limiting Grothendieck ring”  $\mathcal{G}_\infty(\mathcal{C})$  in Section 7, and we establish the structure of this ring in Section 8. We show that the set of basic hooks generates  $\mathcal{G}_\infty(\mathcal{C})$ , and in Section 9 we use partition combinatorics we determine explicitly how certain basis elements of the ring are expressed in terms of the basic hooks. Finally, we discuss some applications to asymptotic representation theory of wreath products in Section 10. In Section 11 we explain how all these results generalise to the setting where  $\mathcal{C}$  is a tensor.

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## 2. PRELIMINARIES

Throughout this paper, we work with a fixed algebraically closed field of characteristic zero,  $k$ , and a category  $\mathcal{C} = \mathcal{R} - \text{mod}$  of finite-dimensional modules for  $\mathcal{R}$ , a fixed Hopf algebra over  $k$ . We work with  $\mathcal{C}$  rather than  $\mathcal{R}$  to stress that our results only depend on the module category. In fact, all our results generalise to the setting where  $\mathcal{C}$  is a tensor category. To accommodate this, we prove things in a suitable level of generality, for example we do not make use of dual modules constructed using the antipode of  $\mathcal{R}$ , which do not exist in a general tensor category.

It is clear that if  $\mathbf{1}$  is the trivial  $\mathcal{R}$  module, then  $\text{End}_{\mathcal{C}}(\mathbf{1}) = k$ . Secondly, the tensor product in  $\mathcal{C}$  is exact in both arguments and bilinear with respect to direct sums. The Grothendieck group,  $\mathcal{G}(\mathcal{C})$ , is the free abelian group generated by isomorphism classes of simple  $\mathcal{R}$ -modules. The exactness of the tensor product implies that it respects the relations of the Grothendieck group and therefore descends to a bilinear distributive multiplication on  $\mathcal{G}(\mathcal{C})$ . Thus,  $\mathcal{G}(\mathcal{C})$ , inherits the structure of a ring. This will be the main setting in which we work.

### 3. PARTITIONS AND SYMMETRIC FUNCTIONS

We will make considerable use of partition combinatorics, which we review briefly. All the material that we will need can be found in the first chapter of [Mac95].

**3.1. Partitions, Symmetric functions, Representations of Symmetric Groups.** We say that  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$  is a partition of  $n$  if  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l$  are nonnegative integers summing to  $n$ , and we call the  $\lambda_i$  the parts of the partition. Two partitions that differ only by the number of trailing zeroes are considered equivalent. The set of all partitions is denoted  $\mathcal{P}$ . We write  $\lambda \vdash n$  to mean that  $\lambda$  is a partition of  $n$ . An alternative way of expressing  $\lambda$  is  $(1^{m_1} 2^{m_2} \dots r^{m_r})$ , where  $m_i$  is the number of  $j$  such that  $\lambda_j = i$ ; in case it is unclear which partition we are considering, we write  $m_i(\lambda)$  for the number of parts of  $\lambda$  equal to  $i$ . The size of  $\lambda$  is  $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_k = 1m_1 + 2m_2 + \dots + rm_r$  (where  $r = \lambda_1$  is the largest part of  $\lambda$ ), which is the unique integer  $n$  such that  $\lambda \vdash n$ . The length  $l(\lambda)$  is the number of nonzero parts of  $\lambda$ , so we have  $l(\lambda) = m_1 + m_2 + \dots + m_r$ . If  $\lambda^{(j)}$  are partitions, we write  $\cup_j \lambda^{(j)}$  for the partition  $\mu$  obtained by merging all the partitions  $\lambda^{(j)}$  together, so  $m_i(\mu) = \sum_j m_i(\lambda^{(j)})$ . We write  $\varepsilon(\lambda) = (-1)^{|\lambda| - l(\lambda)}$ , and  $z_\lambda = (m_1!) 1^{m_1} (m_2!) 2^{m_2} \dots (m_n!) n^{m_n}$ .

Recall that the ring of symmetric functions,  $\Lambda$  is defined as a (graded) inverse limit of the rings  $\mathbb{Z}[x_1, x_2, \dots, x_n]^{S_n}$ , where the symmetric group acts by permuting the variables. It is freely generated as a polynomial algebra by the elementary symmetric functions  $e_i$ , but also by the complete symmetric functions  $h_i$ , so  $\Lambda = \mathbb{Z}[e_1, e_2, \dots] = \mathbb{Z}[h_1, h_2, \dots]$ . There are also power-sum symmetric functions  $p_n$  which do not generate  $\Lambda$  over  $\mathbb{Z}$ , but do satisfy  $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[p_1, p_2, \dots]$ . If we define  $E(t) = \sum_{n=0}^{\infty} e_n t^n$ ,  $H(t) = \sum_{n=0}^{\infty} h_n t^n$ , and  $P(t) = \sum_{n=0}^{\infty} p_{n+1} t^n$ , then we have the relations  $H(t)E(-t) = 1$ , and  $\frac{E'(t)}{E(t)} = P(-t)$ . For a partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ , we write  $e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots e_{\lambda_k}$ , and similarly we define  $h_\lambda$  and  $p_\lambda$  (so the  $e_\lambda$  and  $h_\lambda$  form  $\mathbb{Z}$ -bases of  $\Lambda$ ). Another important family of symmetric functions are the Schur functions  $s_\lambda$  (indexed by  $\lambda \in \mathcal{P}$ ), which form a  $\mathbb{Z}$ -basis of  $\Lambda$ .

The irreducible representations of the symmetric group  $S_n$  in characteristic zero are indexed by partitions  $\lambda \vdash n$ . They are called Specht modules and are denoted by  $S^\lambda$ . Since the conjugacy classes of  $S_n$  are also parametrised by partitions of  $n$  via cycle type, we may write  $\chi_\mu^\lambda$  for the value of the character of  $S^\lambda$  on an element of cycle type  $\mu$ . This allows us to express the Schur function  $s_\lambda$  in terms of power-sum symmetric function as follows:

$$s_\lambda = \sum_{\mu \vdash \lambda} \frac{\chi_\mu^\lambda p_\mu}{z_\mu}$$

Since  $h_n = s_{(n)}$  and  $e_n = s_{(1^n)}$ , we have:

$$h_n = \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda}, \quad e_n = \sum_{\lambda \vdash n} \frac{\varepsilon_\lambda p_\lambda}{z_\lambda}$$

There is a nondegenerate bilinear form  $\langle -, - \rangle$  on  $\Lambda$  for which the Schur functions are orthonormal. It satisfies  $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda, \mu} z_\lambda$ , where  $\delta_{\lambda, \mu}$  is the Kronecker delta.

**3.2. Sets of Variables for Symmetric Functions and Some Stability Properties.** One can think of elements of  $\Lambda \otimes \Lambda$  as symmetric functions that are symmetric in two sets of variables separately. We write  $f(\underline{x})$  to indicate that  $f$  is a symmetric function in the set of variables  $\{x_i\}$  (we will suppress the index set of the variables), or  $f(\underline{x}, \underline{y})$  to mean that  $f$  is a symmetric function in the set of variables  $\{x_i\} \cup \{y_j\}$ . Similarly, we write  $f(\underline{xy})$  when the variable set is  $\{x_i y_j\}$  (for example,  $p_n(\underline{x}, \underline{y}) = p_n(\underline{x}) + p_n(\underline{y})$  and  $p_n(\underline{xy}) = p_n(\underline{x}) p_n(\underline{y})$ ). With this in mind, we have the formal power series identity:

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_{\lambda \in \mathcal{P}} s_\lambda(\underline{x}) s_\lambda(\underline{y}) = \sum_{\mu \in \mathcal{P}} \frac{p_\mu(\underline{x}) p_\mu(\underline{y})}{z_\mu}$$

We also note that  $s_\lambda(\underline{x}\underline{y}) = \sum_{\mu, \nu \in \mathcal{P}} k_{\mu, \nu}^\lambda s_\mu(\underline{x}) s_\nu(\underline{y})$ , where  $k_{\mu, \nu}^\lambda$  are the Kronecker coefficients, defined for  $|\lambda| = |\mu| = |\nu|$  as multiplicities in tensor products of Specht modules (and taken to be zero otherwise):

$$S^\mu \otimes S^\nu = \bigoplus_{\lambda} (S^\lambda)^{\oplus k_{\mu, \nu}^\lambda}$$

On the other hand,  $s_\lambda(\underline{x}, \underline{y}) = \sum_{\mu, \nu} c_{\mu, \nu}^\lambda s_\mu(\underline{x}) s_\nu(\underline{y})$ , where  $c_{\mu, \nu}^\lambda$  are the Littlewood-Richardson coefficients, which also satisfy the property that  $s_\mu s_\nu = \sum_{\lambda} c_{\mu, \nu}^\lambda s_\lambda$ . The Littlewood-Richardson coefficient  $c_{\mu, \nu}^\lambda$  is taken to be zero if  $|\mu| + |\nu| \neq |\lambda|$ . The Kronecker coefficients famously satisfy the following stability property.

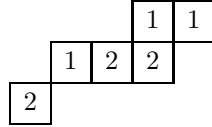
**Lemma 3.1.** *For a partition  $\lambda$ , let  $\tilde{\lambda}(n) = (n - |\lambda|, \lambda)$  be the partition of  $n$  obtained adding one row of length  $n - |\lambda|$  to  $\lambda$  (we assume  $n - |\lambda| \geq \lambda_1$ ). Then, the sequence of Kronecker coefficients  $k_{\tilde{\mu}(n), \tilde{\nu}(n)}^{\tilde{\lambda}(n)}$  eventually becomes constant, and the stable limit, called the reduced Kronecker coefficient, is denoted  $\tilde{k}_{\mu, \nu}^\lambda$ .*

The following result of [Dvi93] shows that in special cases, more can be said.

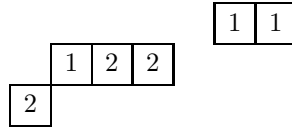
**Lemma 3.2.** *If  $|\mu| + |\nu| = |\lambda|$ , then  $\tilde{k}_{\mu, \nu}^\lambda = c_{\mu, \nu}^\lambda$ .*

**Proposition 3.3.** *Suppose that for a partition  $\lambda$  we write  $\lambda^{*m}$  for the partition obtained by adding  $m$  to  $\lambda_1$ . Then if  $\lambda, \mu, \nu$  are partitions, then the sequence of Littlewood-Richardson coefficients  $c_{\mu, \nu}^\lambda, c_{\mu^{*1}, \nu}^{\lambda^{*1}}, c_{\mu^{*2}, \nu}^{\lambda^{*2}}, \dots$  is eventually constant.*

*Proof.* Using the Littlewood-Richardson rule, it suffices to count the number of skew tableaux of shape  $\lambda^{*m}/\mu^{*m}$  and weight  $\nu$  satisfying the lattice word condition. It is easy to see that the diagrams are related for successive  $m$  by shifting the first row. We illustrate this with an example. Suppose  $\lambda = (5, 4, 1)$ ,  $\mu = (3, 1)$  and  $\nu = (3, 3)$ . An example of a skew-tableau of shape  $\lambda/\mu$  and weight  $\nu$  is



But when  $\lambda^{*2} = (7, 4, 1)$ ,  $\mu^{*2} = (5, 1)$  and  $\nu = (3, 3)$  an example of a skew-tableau of shape  $\lambda^{*2}/\mu^{*2}$  and weight  $\nu$  is



It is clear that as soon as  $m$  is large enough that the first row of the skew diagram of shape  $\lambda^{*m}/\mu^{*m}$  is disconnected from the rest of the diagram then the operation of further shifting the top row to the right leads to an obvious bijection of skew tableaux (the disconnected condition guarantees that the tableau property is unaffected by this operation). Since it also preserves the lattice words associated to the tableaux, the lattice word property is also preserved by this bijection. Counting the number of such tableaux gives the Littlewood-Richardson coefficient  $c_{\mu^{*m}, \nu}^{\lambda^{*m}}$ , which gives the result.  $\square$

#### 4. WREATH PRODUCTS

We outline some features of wreath products that are important for us.

#### 4.1. Construction of $\mathcal{W}_n(\mathcal{C})$ and Restriction/Induction.

**Definition 4.1.** Recall that  $\mathcal{R} \wr S_n$  is the Hopf algebra isomorphic to  $\mathcal{R}^{\otimes n} \otimes kS_n$  as a vector space, with multiplication defined as follows. Suppose that  $a_i$  and  $b_i$ ,  $i \in \{1, 2, \dots, n\}$  are elements of  $\mathcal{R}$ , whilst  $\sigma$  and  $\rho$  are elements of  $S_n$ . Then:

$$((a_1 \otimes a_2 \otimes \dots \otimes a_n) \otimes \sigma) ((b_1 \otimes b_2 \otimes \dots \otimes b_n) \otimes \rho) = (a_1 b_{\sigma^{-1}(1)} \otimes a_2 b_{\sigma^{-1}(2)} \otimes \dots \otimes a_n b_{\sigma^{-1}(n)}) \otimes (\sigma \rho)$$

It is well known that this algebra naturally inherits the structure of a Hopf algebra from the Hopf algebra structure on  $\mathcal{R}$ . The wreath product category  $\mathcal{W}_n(\mathcal{C})$  is the category  $(\mathcal{R} \wr S_n) - \text{mod}$ . We suppress  $\mathcal{R}$  in the notation because  $\mathcal{W}_n(\mathcal{C})$  can be constructed from  $\mathcal{C} = \mathcal{R} - \text{mod}$  alone (see the final section for details).

In our situation, it will be important to consider actions of subgroups of  $S_n$ . In the above, we may form the Hopf subalgebra  $\mathcal{R}^{\otimes n} \otimes kG$  for any subgroup  $G$  of  $S_n$ . If  $H$  is a subgroup of  $G$ , we have a restriction functor  $\text{Res}_H^G : \mathcal{R}^{\otimes n} \otimes kG \rightarrow \mathcal{R}^{\otimes n} \otimes kH$ . Additionally there is an induction functor  $\text{Ind}_H^G : \mathcal{R}^{\otimes n} \otimes kH \rightarrow \mathcal{R}^{\otimes n} \otimes kG$  which is both right adjoint and left adjoint to  $\text{Res}_H^G$ . The induction functor may be written as a sum over coset representatives of  $H$  in  $G$  as follows:

$$\text{Ind}_H^G(M) = \bigoplus_{g \in G/H} gM$$

In the above formula,  $gM$  denotes an object isomorphic to  $M$  as a vector space, and the action of  $G$  is that of an induced representation. Explicitly, to see how  $x \in G$  acts on  $gM$ , note that  $xg = g'h$  for unique coset representative  $g' \in G/H$  and  $h \in H$ . Then,  $x$  takes  $gM$  to  $g'M$ , whilst acting by the usual action of  $h \in H$ . It is clear that induction and restriction define homomorphisms between the relevant Grothendieck groups. We will be interested in the case where  $G = S_n$  and  $H$  is a subgroup of the following type.

**Definition 4.2.** If  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$  is a composition of  $n$  (that is, a finite sequence of nonnegative integers summing to  $n$ ), then  $S_\alpha = \prod_{i=1}^k S_{\alpha_i}$  is the Young subgroup of  $S_n$  corresponding to the composition  $\alpha$  (it is realised as a subgroup in the obvious way). We refer to the factors  $S_{\alpha_i}$  as the component groups of  $S_\alpha$ .

Note that for  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ , we may identify  $\mathcal{R}^{\otimes n} \otimes kS_\alpha$  with  $\bigotimes_i (\mathcal{R}^{\otimes \alpha_i} \otimes kS_{\alpha_i}) = \bigotimes_i (\mathcal{R} \wr S_{\alpha_i})$ .

**4.2. Description of Simple Objects in  $\mathcal{W}_n(\mathcal{C})$ .** In the sequel we will often consider objects of the following form.

**Definition 4.3.** Let  $M \in \mathcal{C}$ , and  $V$  be a finite-dimensional representation of  $S_n$  over  $k$ . We define the following object of  $\mathcal{W}_n(\mathcal{C})$ .

$$M^{\boxtimes n} \otimes V$$

This has a  $\mathcal{R}^{\otimes n}$ -action by acting on the first tensor factor. An element of  $S_n$  acts by permuting the factors of  $M^{\boxtimes n}$  in the obvious way, as well as acting on  $V$ .

We now introduce some standard properties of these objects. The proofs of most of these statements are well known, and therefore omitted.

**Proposition 4.4.** Let  $V_1, V_2$  be finite-dimensional representations of  $S_n$ , and let  $M, M'$  be objects of  $\mathcal{C}$ . We have the following.

(1)

$$M^{\boxtimes n} \otimes (V_1 \oplus V_2) = (M^{\boxtimes n} \otimes V_1) \oplus (M^{\boxtimes n} \otimes V_2)$$

(2)

$$(M^{\boxtimes n} \otimes V_1) \otimes (M'^{\boxtimes n} \otimes V_2) = (M \otimes M')^{\boxtimes n} \otimes (V_1 \otimes V_2)$$

When considering Mackey theory, it will also be necessary to understand the behaviour of  $M^{\boxtimes n} \otimes V$  under induction and restriction.

**Proposition 4.5.** (1) Suppose that  $M$  is an object of  $\mathcal{C}$ ,  $V_1$  is a finite-dimensional representation of  $S_{n_1}$ , and  $V_2$  is a finite-dimensional representation of  $S_{n_2}$ . Then:

$$\text{Ind}_{S_{n_1} \times S_{n_2}}^{S_{n_1+n_2}} ((M^{\boxtimes n_1} \otimes V_1) \boxtimes (M^{\boxtimes n_2} \otimes V_2)) = M^{\boxtimes (n_1+n_2)} \otimes \text{Ind}_{S_{n_1} \times S_{n_2}}^{S_{n_1+n_2}} (V_1 \boxtimes V_2)$$

- (2) Suppose that  $M$  is an object of  $\mathcal{C}$ , and  $V$  is a finite-dimensional representation of  $S_n$  such that  $\text{Res}_{S_{n_1} \times S_{n_2}}^{S_n}(V) = \bigoplus_i V_1^{(i)} \boxtimes V_2^{(i)}$  (where  $V_j^{(i)}$  is a representation of  $S_{n_j}$  for  $j = 1, 2$ ). Then:

$$\text{Res}_{S_{n_1} \times S_{n_2}}^{S_n}(M^{\boxtimes n} \otimes V) = \bigoplus_i (M^{\boxtimes n_1} \otimes V_1^{(i)}) \boxtimes (M^{\boxtimes n_2} \otimes V_2^{(i)})$$

Eventually, we will pass to the Grothendieck group of  $\mathcal{W}_n(\mathcal{C})$ , and we will wish to understand the composition factors of  $M^{\boxtimes n} \otimes V$ . The following proposition will allow us to calculate the composition factors that we will be interested in. The proof is routine, and we omit it.

**Proposition 4.6.** *Denote the image of an object  $R$  in the Grothendieck group of  $\mathcal{W}_n(\mathcal{C})$  by  $[R]$ . Suppose  $N$  is a subobject of  $M$ . If  $\mathbf{1}_G$  denotes the trivial representation of a group  $G$ , we have the following equality in  $\mathcal{G}(\mathcal{W}_n(\mathcal{C}))$ :*

$$[M^{\boxtimes n} \otimes \mathbf{1}_{S_n}] = \sum_{r=0}^n [\text{Ind}_{S_r \times S_{n-r}}^{S_n} ((N^{\boxtimes r} \otimes \mathbf{1}_{S_r}) \boxtimes ((M/N)^{\boxtimes (n-r)} \otimes \mathbf{1}_{S_{n-r}}))]$$

We now describe the simple objects in the category  $\mathcal{W}_n(\mathcal{C})$ . Let  $I(\mathcal{C})$  denote the set of isomorphism classes of simple objects of  $\mathcal{C}$ , and as in [Har16], let:

$$\mathcal{P}_n^{\mathcal{C}} = \{\lambda : I(\mathcal{C}) \rightarrow \mathcal{P} \mid \sum_{U \in I(\mathcal{C})} |\lambda(U)| = n\}$$

Thus,  $\mathcal{P}_n^{\mathcal{C}}$  is the set of multipartitions of  $n$  whose constituent partitions are indexed by isomorphism classes of simple objects in  $\mathcal{C}$ . The set  $\mathcal{P}_n^{\mathcal{C}}$  gives an index set for the isomorphism classes of simple objects of  $\mathcal{W}_n(\mathcal{C})$ .

**Definition 4.7.** *Let  $\lambda \in \mathcal{P}_n^{\mathcal{C}}$ , and  $K = \prod_{U \in I(\mathcal{C})} S_{|\lambda(U)|}$ , a Young subgroup of  $S_n$ . We define  $R_\lambda$ , an object of  $\mathcal{W}_n(\mathcal{C})$ :*

$$R_\lambda = \text{Ind}_K^{S_n} \left( \bigotimes_{U \in I(\mathcal{C})} (U^{\boxtimes |\lambda(U)|} \otimes S^{\lambda(U)}) \right)$$

As before,  $S^\mu$  denotes the Specht module associated to an integer partition  $\mu$ .

These are the (pairwise non isomorphic) simple objects of  $\mathcal{W}_n(\mathcal{C})$ . In [Mor12], this is shown in the context of indecomposable objects of an additive category, but the proof in this setting is analogous. We will use Mackey theory to calculate tensor products of these, and hence the multiplicative structure of the Grothendieck ring  $\mathcal{G}(\mathcal{W}_n(\mathcal{C}))$ .

## 5. MACKEY THEORY

The following results are completely analogous to the corresponding versions for representations of finite groups, including the proofs, which we omit.

**Proposition 5.1.** *Given a finite group  $G$  with subgroups  $H$  and  $K$ , let  $S$  be a collection of  $(H, K)$ -double coset representatives in  $G$ . Suppose that  $M$  is a  $K$ -equivariant object of a  $k$ -linear abelian category with a  $G$ -action. We have the following formula for the composition of induction and restriction.*

$$\text{Res}_H^G(\text{Ind}_K^G(M)) \cong \bigoplus_{s \in S} \text{Ind}_{H \cap s K s^{-1}}^H (\text{Res}_{H \cap s K s^{-1}}^{s K s^{-1}}(sM))$$

**Lemma 5.2.** *If  $H$  is a subgroup of the finite group  $G$ , and  $M, N$  are objects of a monoidal  $k$ -linear abelian category such that  $M$  is  $H$ -equivariant and  $N$  is  $G$ -equivariant, then we have  $\text{Ind}_H^G(M) \otimes N \cong \text{Ind}_H^G(M \otimes \text{Res}_H^G(N))$ . Similarly  $N \otimes \text{Ind}_H^G(M) \cong \text{Ind}_H^G(\text{Res}_H^G(N) \otimes M)$*

**Proposition 5.3.** *Suppose that  $H$  and  $K$  are subgroups of the finite group  $G$ . If  $M$  is an  $H$ -equivariant object of a monoidal  $k$ -linear abelian category with an action of  $G$ , and  $N$  is a  $K$ -equivariant object, and  $S$  is a collection of  $(H, K)$ -double cosets in  $G$ , we have the following.*

$$\text{Ind}_H^G(M) \otimes \text{Ind}_K^G(N) = \bigoplus_{s \in S} \text{Ind}_{H \cap s K s^{-1}}^G (\text{Res}_{H \cap s K s^{-1}}^H(M) \otimes \text{Res}_{H \cap s K s^{-1}}^{s K s^{-1}}(sN))$$

*Proof.*

$$\begin{aligned}
\text{Ind}_H^G(M) \otimes \text{Ind}_K^G(N) &= \text{Ind}_H^G(M \otimes \text{Res}_H^G(\text{Ind}_K^G(N))) \\
&= \text{Ind}_H^G(M \otimes \bigoplus_{s \in S} \text{Ind}_{H \cap sKs^{-1}}^H(\text{Res}_{H \cap sKs^{-1}}^{sKs^{-1}}(sN))) \\
&= \bigoplus_{s \in S} \text{Ind}_H^G(M \otimes \text{Ind}_{H \cap sKs^{-1}}^H(\text{Res}_{H \cap sKs^{-1}}^{sKs^{-1}}(sN))) \\
&= \bigoplus_{s \in S} \text{Ind}_H^G(\text{Ind}_{H \cap sKs^{-1}}^H(\text{Res}_{H \cap sKs^{-1}}^H(M) \otimes \text{Res}_{H \cap sKs^{-1}}^{sKs^{-1}}(sN))) \\
&= \bigoplus_{s \in S} \text{Ind}_{H \cap sKs^{-1}}^G(\text{Res}_{H \cap sKs^{-1}}^H(M) \otimes \text{Res}_{H \cap sKs^{-1}}^{sKs^{-1}}(sN))
\end{aligned}$$

Here we have used the transitivity of induction, namely that  $\text{Ind}_H^G \circ \text{Ind}_{H \cap sKs^{-1}}^H = \text{Ind}_{H \cap sKs^{-1}}^G$  (the proof of this is again analogous to the proof for representations of finite groups).  $\square$

In our setting,  $H$  and  $K$  will be Young subgroups of  $S_n$  (recall that the simple objects of the wreath product category are obtained by applying induction functors from Young subgroups). So if we are to use the previous proposition to decompose tensor products of simple objects, it will be important to understand double coset representatives of Young subgroups of  $S_n$ .

## 6. DOUBLE COSETS OF YOUNG SUBGROUPS

We now prove some facts about minimal length double coset representatives of Young subgroups of symmetric groups. Let  $\sigma \in S_n$  be considered as a bijective function from the set  $\{1, 2, \dots, n\}$  to itself. If  $\mu = (\mu_1, \mu_2, \dots)$  and  $\nu = (\nu_1, \nu_2, \dots)$  are compositions of  $n$  and  $S_\mu = \prod_i S_{\mu_i}$ ,  $S_\nu = \prod_i S_{\nu_i}$  are the associated Young subgroups of  $S_n$ , we seek to describe the  $(S_\mu, S_\nu)$ -double cosets of  $S_n$  as follows. We write  $A_i$  for the subset of  $\{1, 2, \dots, n\}$  that is permuted by  $S_{\mu_i}$  (considered as a subgroup of  $S_\mu$ ), so that  $A_1 = \{1, 2, \dots, \mu_1\}$ ,  $A_2 = \{\mu_1 + 1, \mu_1 + 2, \dots, \mu_1 + \mu_2\}$  and so on. Similarly we define  $B_i$  to be the subsets of  $\{1, 2, \dots, n\}$  permuted by the  $S_{\nu_i}$ .

**Definition 6.1.** Say that  $\sigma \in S_n$  fully ordered if (in the notation defined above), for each  $B_i$  and  $A_j$ , the restrictions  $\sigma|_{B_i}$  and  $\sigma^{-1}|_{A_j}$  are monotone increasing functions.

**Remark 6.2.** The property of being fully ordered will turn out to be equivalent to being a minimal length  $(S_\mu, S_\nu)$ -double coset representative. However, it will be more convenient to work with the above definition.

**Lemma 6.3.** The natural numbers  $C_{i,j}(\sigma) = |\{x \in B_i | \sigma(x) \in A_j\}|$  are double coset invariants. Moreover,  $\sigma_1$  and  $\sigma_2$  are in the same double coset if and only if  $C_{i,j}(\sigma_1) = C_{i,j}(\sigma_2)$  for all  $i, j$ . Additionally, each double coset has a unique fully ordered element.

*Proof.* It is clear that the  $C_{i,j}(\sigma)$  are constant on each double coset since both the left and right actions preserve each  $A_j$  and  $B_i$ . We show that every element can be acted on by the left by  $S_\mu$  and on the right by  $S_\nu$  to obtain a totally ordered element. Then we show that a totally ordered element is determined by the  $C_{i,j}(\sigma)$ . This implies that if two elements of  $S_n$  have the same  $C_{i,j}(\sigma)$ , then they have the same fully ordered element in their double coset, implying that they are in the same coset.

Given  $\sigma \in S_n$ , we may act on it on the right by elements of the  $S_{\nu_i}$  (and hence an element of  $S_\nu$ ) to reorder the elements of  $B_i$  in order of increasing image under  $\sigma$ . This gives a new  $\sigma \in S_n$  such that if  $x < y$  are elements of  $B_i$ , then  $\sigma(x) < \sigma(y)$ . This means that the restriction of  $\sigma$  to each  $B_i$  is a monotone increasing function. We may also act on the left by the  $S_{\mu_i}$  so that the following property holds. If  $x \in B_a$  and  $y \in B_b$  with  $a < b$ , such that  $\sigma(x)$  and  $\sigma(y)$  are in  $A_i$ , then  $\sigma(x) < \sigma(y)$ . Note that this process preserves the property that  $\sigma$  is monotone increasing when restricted to the  $B_i$ . Thus the result of this process is a fully ordered element.

Next we inductively construct a fully ordered  $\sigma$  from prescribed  $C_{i,j}$ . It is easy to see that for a collection of natural numbers  $C'_{i,j}$ , there is a  $\sigma \in S_n$  such that  $C_{i,j}(\sigma) = C'_{i,j}$  if and only if the following two

conditions hold. For each  $j$ ,  $\sum_i C'_{i,j} = |B_j|$  and for each  $i$ ,  $\sum_j C'_{i,j} = |A_i|$ . The first  $C_{1,1}(\sigma)$  elements of  $B_1$  must map to the first  $C_{1,1}(\sigma)$  elements of  $A_1$  (in a monotone increasing way, hence uniquely). Then, the next  $C_{1,2}(\sigma)$  elements of  $B_1$  map to the first  $C_{1,2}(\sigma)$  elements of  $A_2$ , and so on. This means that the image of  $B_1$  is determined uniquely. Then, the first  $C_{2,1}(\sigma)$  elements of  $B_2$  map to the next  $C_{2,1}(\sigma)$  elements of  $A_1$ , and so on (again without choice). Repeating for all  $i$  and  $j$ , we obtain a fully ordered element  $\sigma$  and this process makes it clear that it is unique.  $\square$

**Remark 6.4.** *Noting that the length of  $\sigma \in S_n$  is equal to the number of inversions (that is, pairs  $(i, j)$  with  $1 \leq i < j \leq n$  such that  $\sigma(j) < \sigma(i)$ ), it is easy to check that the property of being fully ordered is the same as being a minimal length double coset representative.*

For convenience, in this section we require that for a composition  $\alpha$ , the factors in the Young subgroup  $S_\alpha = \prod_i S_{\alpha_i}$  are ordered in increasing order from left to right. For example,  $S_{(3,2,1)} = S_1 \times S_2 \times S_3$ . We will be interested in the operation of increasing the largest part of a partition by 1 (hence passing from partitions of  $n$  to partitions of  $n+1$ ). More generally, if  $\alpha$  is a composition of  $n$ , write  $\alpha^*$  for composition of  $n+1$  obtained by adding 1 to the first part of  $\alpha$ . Correspondingly, we discuss  $(S_\mu, S_\nu)$ -double coset representatives under the operation of adding 1 to the largest parts of the partitions  $\mu$  and  $\nu$ . It is easy to see that if  $f$  is a bijection from the set  $\{1, 2, \dots, n\}$  to itself satisfying the fully ordered property, it continues to satisfy the fully ordered property as a function on  $\{1, 2, \dots, n+1\}$  when we define  $f(n+1) = n+1$  (note that this corresponds to the inclusion  $S_n \hookrightarrow S_{n+1}$  by considering elements fixing  $n+1$ ).

**Proposition 6.5.** *Let  $\mu, \nu$  be partitions of  $n$ . After sufficiently many repeated applications of the operation  $(\mu, \nu) \mapsto (\mu^*, \nu^*)$ , the number of  $(S_\mu, S_\nu)$ -double cosets in  $S_n$  stabilises. Moreover, one can choose representatives which are identified for different  $n$  (sufficiently large) via the usual inclusions of symmetric groups.*

*Proof.* Observe that if the largest parts of  $\mu^*$  and  $\nu^*$  each exceed  $n/2$ , then  $C_{(\mu^*)_1, (\nu^*)_1} \geq 1$  by the pigeonhole principle. This means that for a double coset representative  $\sigma$  satisfying the stated monotonicity properties,  $\sigma(n+1) = n+1$ . In particular, each double coset representative is obtained from a double coset representative of  $(S_\mu, S_\nu)$  under the inclusion of  $S_n$  in  $S_{n+1}$ .  $\square$

**Remark 6.6.** *Ordering the multiplicative factors in the definition of Young subgroup from smallest to largest allows us to take the inclusions  $S_n \hookrightarrow S_{n+1}$  obtained by extending functions on  $\{1, 2, \dots, n\}$  by requiring them to fix  $n+1$ . If we did not do this, we would have to use a nonstandard inclusion. The Young subgroups related by different orderings of their factor groups are conjugate in  $S_n$ , so induced objects coming from the two embeddings are related by a twist which will be irrelevant for our purposes.*

## 7. THE LIMITING GROTHENDIECK RING

We work towards understanding the tensor product, with the aim of constructing a “stable” Grothendieck ring.

**7.1. Tensor Products of Irreducible Modules.** It is clear that  $\mathcal{G}(\mathcal{W}_n(\mathcal{C}))$  naturally inherits the structure of a ring, and has a basis given by  $[R_\lambda]$  for  $\lambda \in \mathcal{P}_\mathcal{C}^n$ .

**Example 7.1.** *If  $\mathcal{C}$  is the category of finite-dimensional vector spaces over  $k$ ,  $\mathcal{G}(\mathcal{W}_n(\mathcal{C}))$  is the representation ring of  $S_n$  over  $k$ .*

Given  $\lambda_1, \lambda_2 \in \mathcal{P}_\mathcal{C}^n$ , we wish to describe  $[R_{\lambda_1}][R_{\lambda_2}] = [R_{\lambda_1} \otimes R_{\lambda_2}]$  as a linear combination of  $[R_\lambda]$ . For this task, we use the categorical Mackey theory results. We write  $H_\lambda = \prod_{U \in I(\mathcal{C})} S_{|\lambda(U)|}$  for the subgroup of  $S_n$  from which  $R_\lambda$  is induced. Recall that  $H_\lambda$  is itself a Young subgroup of  $S_n$ . Let  $S$  be a set of  $(H_{\lambda_1}, H_{\lambda_2})$ -double cosets in  $S_n$ .

$$\begin{aligned}
R_{\lambda_1} \otimes R_{\lambda_2} &= \text{Ind}_{H_{\lambda_1}}^{S_n} \left( \boxtimes_{U \in I(\mathcal{C})} \left( U^{\boxtimes |\lambda_1(U)|} \otimes S^{\lambda_1(U)} \right) \right) \otimes \text{Ind}_{H_{\lambda_2}}^{S_n} \left( \boxtimes_{U \in I(\mathcal{C})} \left( U^{\boxtimes |\lambda_2(U)|} \otimes S^{\lambda_2(U)} \right) \right) \\
&= \bigoplus_{t \in S} \text{Ind}_{H_{\lambda_1} \cap t H_{\lambda_2} t^{-1}}^{S_n} \left( \text{Res}_{H_{\lambda_1} \cap t H_{\lambda_2} t^{-1}}^{H_{\lambda_1}} \left( \boxtimes_{U \in I(\mathcal{C})} \left( U^{\boxtimes |\lambda_1(U)|} \otimes S^{\lambda_1(U)} \right) \right) \right. \\
&\quad \left. \otimes \text{Res}_{H_{\lambda_1} \cap t H_{\lambda_2} t^{-1}}^{t H_{\lambda_2} t^{-1}} \left( t \boxtimes_{U \in I(\mathcal{C})} \left( U^{\boxtimes |\lambda_2(U)|} \otimes S^{\lambda_2(U)} \right) \right) \right)
\end{aligned}
\tag{1}$$



**Remark 7.2.** At this point, we observe that  $H_{\lambda_1} \cap tH_{\lambda_2}t^{-1}$  is a Young subgroup in  $S_n$ , which is a proper subgroup of at least one of  $H_{\lambda_1}$  and  $tH_{\lambda_2}t^{-1}$  unless these two are equal. Later on, this observation will imply the vanishing of certain restrictions of virtual representations. We also note that by the fully ordered property of  $t$ ,  $H_{\lambda_1} \cap tH_{\lambda_2}t^{-1}$  is a Young subgroup of  $S_n$ ; it independently permutes contiguous subsets of  $\{1, 2, \dots, n\}$ .

We now exploit the stability property of double cosets of Young subgroups to define a “limiting Grothendieck group”.

**Definition 7.3.** Let  $\mathbf{1}$  be the unit object of  $\mathcal{C}$ . If  $\lambda \in \mathcal{P}_{\mathcal{C}}^n$ , write  $\lambda^*$  for the element of  $\mathcal{P}_{\mathcal{C}}^{n+1}$  obtained by adding 1 to the largest part of  $\lambda(\mathbf{1})$ . We denote the result of applying this operation  $k$  times by  $\lambda^{*k}$ .

Specifically, we will consider products  $[R_{\lambda_1^{*k}}][R_{\lambda_2^{*k}}]$  for  $\lambda_1, \lambda_2 \in \mathcal{P}_{\mathcal{C}}^n$  (for some  $n$ ) as  $k \rightarrow \infty$ . We first introduce notation to conveniently describe the limit.

**Definition 7.4.** If  $\lambda \in \mathcal{P}_{\mathcal{C}}^n$ , we define the multipartition  $\lambda' \in \mathcal{P}_{\mathcal{C}}^{n-\lambda(\mathbf{1})_1}$  by removing the largest part of the partition  $\lambda(\mathbf{1})$  (if this was the empty partition, it remains the empty partition). Also, if  $\lambda \in \mathcal{P}_{\mathcal{C}}^n$ , write  $\tilde{\lambda}(n)$  for the element of  $\mathcal{P}_{\mathcal{C}}^n$  obtained by appending a part of size  $n - m$  to the start of the partition  $\lambda(\mathbf{1})$ ; this is only defined if  $n - m \geq \lambda(\mathbf{1})_1$ . Explicitly, for all  $U$  different from  $\mathbf{1}$ ,  $\lambda'(U) = \tilde{\lambda}(U) = \lambda(U)$ , and  $\lambda'(\mathbf{1}) = \lambda(\mathbf{1}) \setminus (\lambda(\mathbf{1})_1)$ , whilst for  $n - m \geq \lambda(\mathbf{1})_1$ ,  $\tilde{\lambda}(\mathbf{1}) = (n - m, \lambda(\mathbf{1}))$  (in the obvious notation). We leave the operation undefined if the inequality does not hold. Finally, we define the following set which will index a basis of the limiting Grothendieck ring.

$$\mathcal{P}_{\mathcal{C}} = \{\lambda : I(\mathcal{C}) \rightarrow \mathcal{P} \mid \sum_{U \in I(\mathcal{C})} |\lambda(U)| < \infty\}$$

The above operations satisfy some trivial properties.

**Lemma 7.5.** The operations  $\lambda \mapsto \lambda'$  and  $\lambda \mapsto \tilde{\lambda}(n)$  satisfy the following relations.

- (1)  $\lambda' = \lambda^{*1}$
- (2) If  $\lambda \in \mathcal{P}_{\mathcal{C}}^n$ , then  $\tilde{\lambda}'(n) = \lambda$ .
- (3) If  $\lambda \in \mathcal{P}_{\mathcal{C}}^n$  and  $(n - m) \geq \lambda(\mathbf{1})_1$ , then  $\tilde{\lambda}(n)' = \lambda$ . In particular, this holds for  $n$  sufficiently large.
- (4)  $\cup_{n \in \mathbb{Z}_{\geq 0}} \{\lambda' \mid \lambda \in \mathcal{P}_{\mathcal{C}}^n\} = \mathcal{P}_{\mathcal{C}}$

**Definition 7.6.** Given  $\lambda_1, \lambda_2 \in \mathcal{P}_{\mathcal{C}}$ , we may write (in  $\mathcal{G}(\mathcal{W}_n(\mathcal{C}))$ ) for  $n$  sufficiently large)

$$[R_{\tilde{\lambda}_1(n)}][R_{\tilde{\lambda}_2(n)}] = \sum_{\lambda \in \mathcal{P}_{\mathcal{C}}} c_{\lambda_1, \lambda_2}^{\lambda}(n) [R_{\tilde{\lambda}(n)}]$$

In the above sum, we only consider terms for which  $\tilde{\lambda}(n)$  is well defined. We define the numbers  $c_{\lambda_1, \lambda_2}^{\lambda}(n) \in \mathbb{Z}_{\geq 0}$  in this way, and note that for fixed  $\lambda_1, \lambda_2, \lambda$  it is defined for all sufficiently large  $n$ .

We use Mackey theory to show that the  $c_{\lambda_1, \lambda_2}^{\lambda}(n)$  become constant as  $n \rightarrow \infty$ . Specifically, equation 1 describes the decomposition and Proposition 6.5 implies that the index set of the sum stabilises. So, it suffices to show that for any fixed  $t$  in the set of double coset representatives, the corresponding summand also stabilises:

$$\text{Ind}_{H_{\lambda_1} \cap tH_{\lambda_2}t^{-1}}^{S_n} \left( \text{Res}_{H_{\lambda_1} \cap tH_{\lambda_2}t^{-1}}^{H_{\lambda_1}} \left( \boxtimes_{U \in I(\mathcal{C})} \left( U^{\boxtimes |\lambda_1(U)|} \otimes S^{\lambda_1(U)} \right) \right) \otimes \text{Res}_{H_{\lambda_1} \cap tH_{\lambda_2}t^{-1}}^{tH_{\lambda_2}t^{-1}} \left( t \boxtimes_{U \in I(\mathcal{C})} \left( U^{\boxtimes |\lambda_2(U)|} \otimes S^{\lambda_2(U)} \right) \right) \right)$$

To do this, we first show that the restrictions in the above expression stabilise in a particular sense.

**Lemma 7.7.** Let  $n$  be sufficiently large (depending on  $\lambda_1, \lambda_2$ , and  $t$ ). There exists  $g \in S_n$  (identified for different  $n$  via usual inclusions of symmetric groups) such that  $g(H_{\lambda_1} \cap tH_{\lambda_2}t^{-1})g^{-1} = S_{(n-|\mu|, \mu)}$ , where  $\mu = (\mu_1, \mu_2, \dots, \mu_l)$  is some partition. Additionally, the restriction

$$\text{Res}_{H_{\lambda_1} \cap tH_{\lambda_2}t^{-1}}^{H_{\lambda_1}} \left( \boxtimes_{U \in I(\mathcal{C})} \left( U^{\boxtimes |\lambda_1(U)|} \otimes S^{\lambda_1(U)} \right) \right)$$

is equal to a finite direct sum of expressions of the following form, where the multiplicity of each term does not vary with  $n$ , provided  $n$  is sufficiently large:

$$\left( \boxtimes_{i=1}^l \left( U_i^{\boxtimes |\nu^{(i)}|} \otimes S^{\nu^{(i)}} \right) \right) \boxtimes \left( \mathbf{1}^{\boxtimes (n-|\mu|)} \otimes S^{(n-|\nu^{(0)}|-|\mu|, \nu^{(0)})} \right)$$

Here the  $U_i$  are not necessarily distinct and each  $\nu^{(i)}$  is a partition of  $\mu_i$  ( $\nu^{(0)}$  is arbitrary, but only finitely many cases appear). A similar statement holds for the restriction from  $tH_{\lambda_2}t^{-1}$ .

*Proof.* Firstly, we note that we may instead write the induced representation using a conjugate subgroup, due to the following identity for  $H$  a subgroup of  $G$ ,  $M$  an  $H$ -equivariant object, and  $g \in G$ .

$$\text{Ind}_H^G(M) = g^{-1} \text{Ind}_{gHg^{-1}}^G(gM)$$

This will ultimately allow us to assume that  $H_{\lambda_1} \cap tH_{\lambda_2}t^{-1}$  has the form  $S_{(n-|\mu|, \mu)}$ . The subgroup  $H_{\lambda_1} \cap tH_{\lambda_2}t^{-1}$  is a Young subgroup of  $S_n$  by Remark 7.2, which we may conjugate to reorder the factor groups in order of increasing size. Explicitly, we have  $g \in S_n$  such that  $g(H_{\lambda_1} \cap tH_{\lambda_2}t^{-1})g^{-1} = S_\alpha$  for some partition  $\alpha = (\mu_0, \mu_1, \dots, \mu_l)$ . We now show that we may take  $\mu_1, \mu_2, \dots, \mu_l$  to be constant with respect to  $n$ , and hence  $\mu = (\mu_1, \mu_2, \dots, \mu_l)$  is the desired partition from the statement of the lemma (and  $\mu_0 = n - |\mu|$ ).

We note that since  $t$  was preserved under the inclusions  $S_n \hookrightarrow S_{n+1} \hookrightarrow \dots$ , then  $t(i) = i$  for  $i$  larger than some fixed constant  $m_0$  depending on  $t$ . If  $m_1 = |\lambda_1| - |\lambda_1(\mathbf{1})_1|$ , then by construction  $H_{\lambda_1}$  contains the symmetric group on  $\{m_1 + 1, m_1 + 2, \dots, n\}$  (this is a subgroup of the component group whose representation  $U^{\boxtimes |\nu|} \otimes S^\nu$  has  $U = \mathbf{1}$ ). Similarly if  $m_2 = |\lambda_2| - |\lambda_2(\mathbf{1})_1|$ , then  $H_{\lambda_2}$  contains the symmetric group on  $\{m_2 + 1, m_2 + 2, \dots, n\}$ . Now, we let  $M = \max(m_0, m_1, m_2)$ , and we see that the symmetric group on  $\{M + 1, M + 2, \dots, n\}$  is contained in  $H_{\lambda_1} \cap tH_{\lambda_2}t^{-1}$ . Thus we may choose  $g$  in the previous paragraph to fix  $M + 1, M + 2, \dots, n$ ; this essentially amounts to making the factor group permuting the orbit of  $n$  appear as the last factor in the construction of  $S_\mu$ . By similar reasoning, the other component groups of  $H_{\lambda_1} \cap tH_{\lambda_2}t^{-1}$  are stable when passing from  $n$  to  $n + 1$ , meaning that  $H_{\lambda_1} \cap tH_{\lambda_2}t^{-1}$  decomposes as a product of a fixed number of symmetric groups  $S_{\mu_i}$  (where  $\mu_i$  are constant with respect to  $n$ ) and  $S_{n - \sum_i \mu_i}$ .

Next we discuss stability of the restriction. The restriction of an external product is the same as the external product of restrictions. Omitting the twist by  $g$  for notational clarity, the restriction of interest is:

$$\text{Res}_{H_{\lambda_1} \cap tH_{\lambda_2}t^{-1}}^{H_{\lambda_1}} \left( \boxtimes_{U \in I(\mathcal{C})} \left( U^{\boxtimes |\lambda_1(U)|} \otimes S^{\lambda_1(U)} \right) \right) = \boxtimes_{U \in I(\mathcal{C})} \text{Res}_{S_{|\lambda_1(U)|} \cap tH_{\lambda_2}t^{-1}}^{S_{|\lambda_1(U)|}} \left( U^{\boxtimes |\lambda_1(U)|} \otimes S^{\lambda_1(U)} \right)$$

The twist by  $g$  allows us to assume the form of  $S_{|\lambda_1(U)|} \cap tH_{\lambda_2}t^{-1}$ ; for each  $U$  we will get a Young subgroup of  $S_{|\lambda_1(U)|}$  and the product of these across  $U \in I(\mathcal{C})$  will give  $H_{\lambda_1} \cap tH_{\lambda_2}t^{-1}$ . So we may write  $S_{|\lambda_1(U)|} \cap tH_{\lambda_2}t^{-1} = \prod_{j \in I_U} S_{\mu_j}$ , where the  $I_U$  are disjoint subsets of  $\{0, 1, \dots, l\}$  indexed by  $U \in I(\mathcal{C})$  that partition  $0, 1, \dots, l$ . In particular, when  $U \neq \mathbf{1}$  this is independent of  $n$  (the  $U = \mathbf{1}$  term contains  $S_{\mu_0} = S_{n-|\mu|}$  which does depend on  $n$ ). Let  $c_f \in \mathbb{Z}_{\geq 0}$  be defined by restricting representations of symmetric groups:

$$\text{Res}_{\prod_{i \in I_U} S_{\mu_i}}^{S_{|\lambda_1(U)|}} \left( S^{\lambda_1(U)} \right) = \bigoplus_{f: I_U \rightarrow \mathcal{P}, |f(i)| = \mu_i} \left( \boxtimes_{i \in I_U} S^{f(i)} \right)^{\oplus c_f}$$

After suitably applying Proposition 4.5 to decompose such a restriction, we get the following.

$$\text{Res}_{S_{|\lambda_1(U)|} \cap tH_{\lambda_2}t^{-1}}^{S_{|\lambda_1(U)|}} \left( U^{\boxtimes |\lambda_1(U)|} \otimes S^{\lambda_1(U)} \right) = \bigoplus_{f: I_U \rightarrow \mathcal{P}, |f(i)| = \mu_i} \left( \boxtimes_{i \in I_U} \left( U^{\boxtimes |f(i)|} \otimes S^{f(i)} \right) \right)^{\oplus c_f}$$

This makes it clear that the only dependence on  $n$  enters through the term corresponding to  $\mu_0 = n - |\mu|$  in the  $U = \mathbf{1}$  case. Let  $k = \sum_{i \in I_1 \setminus \{0\}} \mu_i$ . In that case, we observe that restriction from  $S_{|\lambda_1(\mathbf{1})|}$  to  $(\prod_{i \in I_1 \setminus \{0\}} S_{\mu_i}) \times S_{n-|\mu|}$  is the same as first restricting to  $S_k \times S_{n-|\mu|}$  and then restricting the first factor to  $\prod_{i \in I_1 \setminus \{0\}} S_{\mu_i}$  where the latter operation will be independent of  $n$ , similarly to the case  $U \neq \mathbf{1}$ . Thus it is enough to show that the operation of restricting to  $S_k \times S_{n-|\mu|}$  is stable in the sense described by the statement of the lemma.

To understand the restriction, we fix an integer partition  $\rho$ . We must demonstrate the stability of the following expression (understood as a sum of terms of the form  $S^{(n-k-|\nu|, \nu)}$ ).

$$\text{Res}_{S_k \times S_{n-|\mu|}}^{S_{n-|\mu|+k}} (S^{(n-k-|\rho|, \rho)})$$

The restriction multiplicities are given by Littlewood-Richardson coefficients, and the stability condition is immediately implied by Proposition 3.3. The case for the restriction from  $tH_{\lambda_2}t^{-1}$  is completely analogous.  $\square$

Finally, we are able to prove stability of the coefficients  $c_{\lambda_1, \lambda_2}^\lambda(n)$ .

**Theorem 7.8.** *For any choice of  $\lambda_1, \lambda_2, \lambda$ ,  $\lim_{n \rightarrow \infty} c_{\lambda_1, \lambda_2}^\lambda(n)$  exists and is a nonnegative integer.*

*Proof.* We use the previous lemma, again reducing to the case of a fixed double coset representative  $t$ . After performing the restriction, we obtain a finite number of stable summands, so it suffices to show that products of those exhibit the relevant stabilisation property. We write:

$$\begin{aligned} & \text{Ind}_{H_{\lambda_1} \cap tH_{\lambda_2}t^{-1}}^{S_n} \left( \text{Res}_{H_{\lambda_1} \cap tH_{\lambda_2}t^{-1}}^{H_{\lambda_1}} \left( \boxtimes_{U \in I(\mathcal{C})} \left( U^{\boxtimes |\lambda_1(U)|} \otimes S^{\lambda_1(U)} \right) \right) \right. \\ & \quad \left. \otimes \text{Res}_{H_{\lambda_1} \cap tH_{\lambda_2}t^{-1}}^{tH_{\lambda_2}t^{-1}} \left( t \boxtimes_{U \in I(\mathcal{C})} \left( U^{\boxtimes |\lambda_2(U)|} \otimes S^{\lambda_2(U)} \right) \right) \right) \\ &= \bigoplus \text{Ind}_{S_{n-|\mu|} \times S_\mu}^{S_n} \left( \left( \boxtimes_{i=1}^l \left( U_i^{\boxtimes |\sigma^{(i)}|} \otimes S^{\sigma^{(i)}} \right) \right) \boxtimes \left( \mathbf{1}^{\boxtimes (n-|\mu|)} \otimes S^{(n-|\sigma^{(0)}|-|\mu|, \sigma^{(0)})} \right) \right. \\ & \quad \left. \otimes \left( \boxtimes_{i=1}^l \left( V_i^{\boxtimes |\rho^{(i)}|} \otimes S^{\rho^{(i)}} \right) \right) \boxtimes \left( \mathbf{1}^{\boxtimes (n-|\mu|)} \otimes S^{(n-|\rho^{(0)}|-|\mu|, \rho^{(0)})} \right) \right) \\ &= \bigoplus \text{Ind}_{S_{n-|\mu|} \times S_\mu}^{S_n} \left( \left( \boxtimes_{i=1}^l \left( (U_i \otimes V_i)^{\boxtimes |\sigma^{(i)}|} \otimes S^{\sigma^{(i)}} \otimes S^{\rho^{(i)}} \right) \right) \boxtimes \left( \mathbf{1}^{n-|\mu|} \otimes S^{(n-|\sigma^{(0)}|-|\mu|, \sigma^{(0)})} \otimes S^{(n-|\rho^{(0)}|-|\mu|, \rho^{(0)})} \right) \right) \end{aligned}$$

Here the first equality used the statement of the preceding lemma (hence the implied sum is finite and independent of  $n$ );  $\sigma^{(i)}$  and  $\rho^{(i)}$  are the partitions coming from the statement of the lemma. The second equality used Proposition 4.4. Each  $U_i \otimes V_i$  decomposes into a linear combination of the  $[U]$  when we pass to the Grothendieck group. Proposition 4.6 can be used to reduce this to a sum of similar terms where these linear combinations are replaced with  $[U]$  for some  $U \in I(\mathcal{C})$  and the resulting quantity independent of  $n$ . The result that the term  $\mathbf{1}^{\boxtimes (n-|\mu|)} \otimes \left( S^{(n-|\sigma^{(0)}|-|\mu|, \sigma^{(0)})} \otimes S^{(n-|\rho^{(0)}|-|\mu|, \rho^{(0)})} \right)$  admits a stable limit is equivalent to the stability of Kronecker coefficients. Then, taking the exterior tensor product with a finite number of fixed  $\mathbf{1}^{\boxtimes |\nu|} \otimes S^\nu$  (coming from the finite terms in the product) and inducing to a larger symmetric group also has a stable limit because the multiplicities are described by Littlewood-Richardson coefficients which we already know have suitable stability properties as per Proposition 3.3. The final induction allows us to obtain a linear combination  $[R_\lambda]$  in the Grothendieck group. The coefficients are finite because they are stable limits of a sequence of integers.  $\square$

**7.2. Definition and Basic Properties of the Limiting Grothendieck Ring.** We come to the definition of the main object of this paper.

**Definition 7.9.** *Let  $\mathcal{G}_\infty(\mathcal{C})$  be the  $\mathbb{Q}$ -algebra having basis  $X_\lambda$  for  $\lambda \in \mathcal{P}_\mathcal{C}$  and multiplication defined by:*

$$X_{\lambda_1} X_{\lambda_2} = \sum_{\lambda \in \mathcal{P}_\mathcal{C}} \left( \lim_{n \rightarrow \infty} c_{\lambda_1, \lambda_2}^\lambda(n) \right) X_\lambda$$

**Remark 7.10.** *We could have defined  $\mathcal{G}_\infty(\mathcal{C})$  to be a  $\mathbb{Z}$ -algebra instead of a  $\mathbb{Q}$ -algebra, but then certain elements of interest to us would no longer lie in the algebra.*

We also introduce a collection of elements that will be important.

**Definition 7.11.** *A basic hook is an element  $\lambda \in \mathcal{P}_\mathcal{C}$  such that  $\lambda(U) = (1^n)$  for some  $U \in I(\mathcal{C})$ , and  $\lambda(V)$  is the empty partition for all  $V$  different from  $U$ . By abuse of terminology we also refer to  $X_\lambda \in \mathcal{G}_\infty(\mathcal{C})$  as a basic hook whenever the indexing multipartition  $\lambda$  is a basic hook; in this case we also denote it  $e_n(U)$ .*

**Theorem 7.12.** *Asymptotically as  $t \rightarrow \infty$ , the structure constants of the images of indecomposable objects of the Deligne category  $\underline{\text{Rep}}(\mathcal{R} \wr S_t)$  in the relevant Grothendieck group agree with the structure constants of the  $X_\lambda$  in  $\mathcal{G}_\infty(\mathcal{C})$ .*

*Proof.* The wreath product Deligne categories  $\underline{\text{Rep}}(\mathcal{R} \wr S_t)$  admit tensor functors to  $\text{Rep}(\mathcal{R} \wr S_n)$  for  $n \in \mathbb{Z}_{\geq 0}$ . To deduce the result, it is sufficient to have a simple description of how the functor behaves on objects. The object indexed by a multipartition  $\lambda$  is mapped to the irreducible object of  $\text{Rep}(\mathcal{R} \wr S_n)$  indexed by

$\tilde{\lambda}(n)$ , if this multipartition is well defined (i.e.  $n - |\lambda| \geq \lambda(\mathbf{1})_1$ ), and otherwise it is zero. For more details, see Theorem 4.13 and Theorem 5.6 of [Mor12] (see also Theorem 3.1 of [Har16]). In the original setting of  $\text{Rep}(S_t)$ , this was proved by Deligne in [Del07].  $\square$

We have a few preliminary facts about the algebra  $\mathcal{G}_\infty(\mathcal{C})$ .

**Theorem 7.13.** *The algebra  $\mathcal{G}_\infty(\mathcal{C})$  is a unital associative algebra satisfying the following:*

- (1)  $\mathcal{G}_\infty(\mathcal{C})$  is commutative if and only if  $\mathcal{G}(\mathcal{C})$  is commutative.
- (2)  $\mathcal{G}_\infty(\mathcal{C})$  is generated by the basic hooks  $e_n(U)$ , where  $n \geq 1$  and  $U \in I(\mathcal{C})$ .
- (3) There is a filtration  $\mathcal{G}_\infty(\mathcal{C}) = \cup_{n \in \mathbb{N}} \mathcal{F}_n$ , where  $\mathcal{F}_n$  is spanned by  $X_\lambda$  with  $|\lambda| \leq n$ .
- (4) The associated graded algebra of  $\mathcal{G}_\infty(\mathcal{C})$  with respect to this filtration is isomorphic to  $\bigotimes_{U \in I(\mathcal{C})} \Lambda_{\mathbb{Q}}^{(U)}$ , where  $\Lambda_{\mathbb{Q}}^{(U)}$  is the ring of symmetric functions with coefficients in  $\mathbb{Q}$ . If we write  $f^{(U)}$  to indicate that the symmetric function  $f$  is considered as an element of  $\Lambda_{\mathbb{Q}}^{(U)}$ , then the image of  $[R_\lambda]$  is  $\prod_{U \in I(\mathcal{C})} s_{\lambda(U)}^{(U)}$ .

*Proof.* Firstly, the multiplication in  $\mathcal{G}_\infty(\mathcal{C})$  is seen to be associative by considering  $[R_{\tilde{\lambda}_1(n)}][R_{\tilde{\lambda}_2(n)}][R_{\tilde{\lambda}_3(n)}]$  in the associative algebra  $\mathcal{G}(\mathcal{W}_n(\mathcal{C}))$ . For  $n$  sufficiently large, the coefficient of  $[R_{\tilde{\lambda}(n)}]$  in that element becomes equal to the coefficient of  $X_\lambda$  in  $X_{\lambda_1} X_{\lambda_2} X_{\lambda_3}$ , regardless of how the latter product is parenthesised. It is easy to see that the basis element corresponding to the empty partition is the unit element.

The commutativity or non-commutativity of multiplication can be seen from the proof of Theorem 7.8, where (up to conjugation by the double coset representative  $t$ ) the only change between  $[R_{\lambda_1}][R_{\lambda_2}]$  and  $[R_{\lambda_2}][R_{\lambda_1}]$  is the product  $U_i \otimes V_i$  (versus  $V_i \otimes U_i$ ) of objects of  $\mathcal{C}$ .

The filtration is essentially the same as the  $|\lambda|$ -filtration defined in Definition 2.7 of [Har16]. This can also be seen from the proof of 7.8, knowing that the stable Kronecker coefficients  $\tilde{k}_{\mu, \nu}^\lambda$  are zero unless  $|\mu| + |\nu| \geq |\lambda|$ , and when this is an equality, the reduced Kronecker coefficient coincides with the Littlewood-Richardson coefficient corresponding to the same partitions (see Lemma 3.2). This in particular shows that the associated graded algebra (with basis induced from  $X_\lambda$ ) has structure constants equal to those of the ring of symmetric functions with the Schur function basis. In particular under this correspondence, the basic hooks  $e_n(U)$  correspond to elementary symmetric functions  $e_n^{(U)}$  in  $\Lambda_{\mathbb{Q}}^{(U)}$ . This means that the basic hooks generated the associated graded algebra, and hence they generate  $\mathcal{G}_\infty(\mathcal{C})$ .  $\square$

**Example 7.14.** *In the case where  $\mathcal{C} = kG - \text{mod}$  for a finite group  $G$ ,  $\mathcal{W}_n(\mathcal{C})$  is equivalent to the category of finite-dimensional representations for the wreath product  $G^n \rtimes S_n$ . In this case,  $\mathcal{G}_\infty(\mathcal{C})$  is commutative, and it easily follows that  $\mathcal{G}_\infty(\mathcal{C})$  is isomorphic to a free polynomial algebra in the basic hooks. In our setting,  $\mathcal{G}(\mathcal{C})$  may not be commutative, in which case  $\mathcal{G}_\infty(\mathcal{C})$  cannot possibly be a free polynomial algebra. Nevertheless, we will give a description of the algebra structure of the ring in terms of basic hooks, and also give generating functions describing how the  $X_\lambda$  are expressed in terms of basic hooks.*

## 8. THE RING STRUCTURE OF $\mathcal{G}_\infty(\mathcal{C})$

**8.1. The Elements  $T_n(M)$ .** We use the following construction to relate  $\mathcal{W}_n(\mathcal{C})$  with  $\mathcal{G}_\infty(\mathcal{C})$ .

**Definition 8.1.** *Suppose that  $H$  is a subgroup of  $S_n$  and  $M$  is a module over  $\mathcal{R}^{\otimes n} \otimes kH$ . In  $\mathcal{G}(\mathcal{W}_{n+m}(\mathcal{C}))$  we may write*

$$[\text{Ind}_{H \times S_m}^{S_{m+n}} (M \boxtimes (\mathbf{1}^{\boxtimes m} \otimes \mathbf{1}_{S_m}))] = \sum_{\mu} c_{\mu}(M, m) [R_{\tilde{\mu}(n+m)}]$$

Here  $\mathbf{1}_{S_m}$  is the trivial representation of  $S_m$  and for any fixed  $n$  we only sum over  $\mu$  such that  $\tilde{\mu}(n+m)$  is defined. By transitivity of induction, we may replace  $M$  and  $H$  with  $\text{Ind}_H^{S_n}(M)$  and  $S_n$  respectively. In this case, if  $M$  is a simple object, it is induced from something of the form

$$(\mathbf{1}^{\boxtimes |\rho|} \otimes S^\rho) \boxtimes \left( \bigotimes_{U \neq \mathbf{1}} (U^{\boxtimes |\rho(U)|} \otimes S^{\rho(U)}) \right)$$

Substituting this into the previous expression, we see that the stability of Littlewood-Richardson coefficients (Proposition 3.3) implies that there is a nonzero contribution to only finitely many  $c_\mu(M, m)$ , and the contribution becomes constant for  $m$  sufficiently large. We define

$$\lim_{m \rightarrow \infty} \text{Ind}_{H \times S_m}^{S_{m+n}} \left( M \boxtimes \left( \mathbf{1}^{\boxtimes m} \otimes \mathbf{1}_{S_m} \right) \right) = \sum_{\mu \in \mathcal{P}_C} \left( \lim_{m \rightarrow \infty} c_\mu(M, m) \right) X_\mu$$

By the linearity of induction, we may extend this definition to allow  $M$  to be not necessarily simple, or indeed a formal difference of objects.

We now define a generating set of  $\mathcal{G}_\infty(\mathcal{C})$  with favourable multiplicative properties.

**Definition 8.2.** For an object  $M$  of  $\mathcal{C}$  and  $n \in \mathbb{Z}_{>0}$ , we define the following elements of  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{G}(\mathcal{W}_n(\mathcal{C}))$ :

$$T_n^f(M) = \frac{1}{n} \sum_{\lambda \vdash n} \chi_{(n)}^\lambda [M^{\boxtimes |\lambda|} \otimes S^\lambda]$$

We construct an analogous element of  $\mathcal{G}_\infty$  as follows.

$$T_n(M) = \frac{1}{n} \sum_{\lambda \vdash n} \chi_{(n)}^\lambda \lim_{m \rightarrow \infty} \text{Ind}_{S_n \times S_m}^{S_{n+m}} \left( \left( M^{\boxtimes |\lambda|} \otimes S^\lambda \right) \boxtimes \left( \mathbf{1}^{\boxtimes m} \otimes \mathbf{1}_{S_m} \right) \right)$$

We similarly define

$$T_n(M_1, M_2, \dots, M_k) = \lim_{m \rightarrow \infty} \text{Ind}_{S_n^k \times S_m}^{S_{nk+m}} \left( T_n^f(M_1) \boxtimes T_n^f(M_2) \boxtimes \dots \boxtimes T_n^f(M_k) \boxtimes \left( \mathbf{1}^{\boxtimes m} \otimes \mathbf{1}_{S_m} \right) \right)$$

Similar methods show that this is well defined.

**Remark 8.3.** The character orthogonality relation

$$\frac{1}{n} \sum_{\lambda \vdash n} \chi_{(n)}^\lambda \chi_\nu^\lambda = \delta_{\nu, (n)}$$

suggests that one can think of  $T_n(U)$  as a generalisation of the indicator function of cycle type  $(n)$  in a copy of the class functions on  $S_n$  associated to  $U \in I(\mathcal{C})$ . This is based on the fact that the virtual character associated to  $\frac{1}{n} \sum_{\lambda \vdash n} \chi_{(n)}^\lambda S^\lambda$  is the indicator function of  $n$ -cycles on  $S_n$ .

**Proposition 8.4.** We have the following properties of the  $T_n(U)$ , for  $U \in I(\mathcal{C})$ :

- (1) The image of  $T_n(U)$  in the associated graded algebra of  $\mathcal{G}_\infty$ , which we identify with  $\bigotimes_U \Lambda_{\mathbb{Q}}^{(U)}$ , is  $p_n(U)$ . That is, the  $n$ -th power sum symmetric function in  $\Lambda_{\mathbb{Q}}^{(U)}$ .
- (2) Fix a total order on  $\mathbb{Z}_{>0} \times I(\mathcal{C})$ . Consider the monomials in  $T_n(U)$  for  $(n, U) \in \mathbb{Z}_{>0} \times I(\mathcal{C})$  where the factors occur in order consistent with the total order ("PBW monomials"). These monomials are linearly independent.
- (3) The  $T_n(U)$  generate  $\mathcal{G}_\infty$ .
- (4)  $T_n(U)$  lies in the  $n$ -th filtered component of  $\mathcal{G}_\infty$ .

*Proof.* The first claim follows from the fact that the virtual character associated to  $\frac{1}{n} \sum_{\lambda \vdash n} \chi_{(n)}^\lambda S^\lambda$  is the indicator function of  $n$ -cycles on  $S_n$ . The second follows from the fact that the  $p_n(U)$  are algebraically independent in the associated graded algebra. Since the  $p_n(U)$  generated the associated graded algebra, the third claim follows, and the final claim is trivial.  $\square$

The following lemma will underpin much of what follows.

**Lemma 8.5.** Any nontrivial restriction of  $T_n^f(U)$  to a Young subgroup of  $S_n$  is zero.

*Proof.* This immediately follows from the fact that the only Young subgroup of  $S_n$  containing an  $n$ -cycle is all of  $S_n$ .  $\square$

In order to understand the algebra structure of  $\mathcal{G}_\infty(\mathcal{C})$ , we determine the commutator of two elements of the form  $T_n(U)$  (recall that such elements generate the algebra).

**Theorem 8.6.** *Let  $U_1, U_2 \in I(\mathcal{C})$ . If  $n \neq m$ , the commutator of  $T_n(U_1)$  and  $T_m(U_2)$  vanishes:  $[T_n(U_1), T_m(U_2)] = 0$ . If  $N_{U_1, U_2}^{U_3}$  is the structure tensor of the Grothendieck ring (so that  $[U_1][U_2] = \sum_{U_3} N_{U_1, U_2}^{U_3} [U_3]$ ), then we have:*

$$[T_n(U_1), T_m(U_2)] = \sum_{U_3} (N_{U_1, U_2}^{U_3} - N_{U_2, U_1}^{U_3}) T_n(U_3)$$

*Proof.* To calculate the commutator  $[T_n(U_1), T_m(U_2)]$ , we calculate the analogous quantity in  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{G}(\mathcal{W}_{m+n+k}(\mathcal{C}))$  for  $k$  sufficiently large. As each is defined in as the image of an induced object in the Grothendieck group, we may apply the Mackey theory formalism to calculate  $T_n(U_1)T_m(U_2)$ .

Firstly, the number of  $(S_n \times S_{m+k}, S_m \times S_{n+k})$ -double cosets in  $S_{m+n+k}$  is  $\min(m, n) + 1$ . This can be seen from calculating the possible  $C_{i,j}(\sigma)$  that can arise. Both  $i$  and  $j$  may take two different values. Therefore  $C_{1,1}(\sigma)$  determines all other  $C_{i,j}(\sigma)$  via identities such as  $C_{i,1}(\sigma) + C_{i,2}(\sigma) = |B_1| = m$  (similarly  $C_{1,j}(\sigma)$  and  $C_{2,j}(\sigma)$  determine each other). So, double cosets are determined by a single invariant  $C_{1,1}(\sigma)$ , namely, the number of elements of  $\{1, 2, \dots, m\}$  that are mapped to the set  $\{1, 2, \dots, n\}$ . Clearly this number can take any of the values  $0, 1, \dots, \min(m, n)$ .

Consider a double coset representative  $\sigma$  (interpreted as a bijection from the set  $\{1, 2, \dots, n+m+k\}$  to itself) such that  $f(\{1, 2, \dots, n\}) \neq \{1, 2, \dots, m\}$  and  $f(\{1, 2, \dots, n\}) \cap \{1, 2, \dots, m\} \neq \emptyset$ . In the Mackey theoretic computation the summand coming from a twist by  $\sigma$  will involve restricting to  $(S_n \times S_{m+k}) \cap \sigma(S_m \times S_{n+k})\sigma^{-1}$ , which will not contain the entirety of  $S_n$  (considered as a subgroup of  $S_n \times S_{m+k}$ ). In particular, the calculation involves restricting  $T_n(U)$  to a proper Young subgroup of  $S_n$ , giving zero by Lemma 8.5. There are two cases that need to be considered;  $C_{1,1}(\sigma) = 0$  and  $C_{1,1}(\sigma) = n = m$ . The first case gives rise to the following term:

$$\text{Ind}_{S_n \times S_m \times S_k}^{S_{n+m+k}} (T_n^f(U_1) \boxtimes T_m^f(U_2) \boxtimes [\mathbf{1}^{\boxtimes k} \otimes \mathbf{1}_{S_k}])$$

Since  $S_n \times S_m \times S_k$  and  $S_m \times S_n \times S_k$  are conjugate subgroups of  $S_{m+n+k}$ , it follows that if  $T_n^f(U_1)$  and  $T_m^f(U_2)$  were interchanged, the contribution from the corresponding term would be the same, in particular, the contribution of this term to the commutator is zero. In particular, if  $n \neq m$ ,  $[T_n(U_1), T_m(U_2)] = 0$ .

If  $n = m$ , then there we consider the contribution from the double coset representative which identifies the symmetric group factors associated to  $S_n$  and  $S_m$  in the respective Young subgroups. Working in  $\mathcal{G}(\mathcal{W}_n(\mathcal{C}))$  and applying Proposition 4.4 we find:

$$T_n^f(U_1)T_n^f(U_2) = T_n^f(U_1 \otimes U_2)$$

Here we used the fact that the indicator function of  $n$ -cycles (considered as a class function on  $kS_n$ ) is an idempotent for the tensor product. We may use Proposition 4.6 to express  $T_n(U_1 \otimes U_2)$  in terms of  $T_n(U)$  for  $U \in I(\mathcal{C})$ . The sum in Proposition 4.6 multiplied by  $T_n^f(\mathbf{1})$  reduces to the first and last term (the other terms involve a nontrivial restriction of  $T_n^f(\mathbf{1})$  to a Young subgroup); we get  $T_n(M) = T_n(N) + T_n(M/N)$ . Iterating this, we get one term for each composition factor of  $U_1 \otimes U_2$ . If  $N_{U_1, U_2}^{U_3}$  is the structure tensor of the Grothendieck ring, then we have:

$$[T_n(U_1), T_n(U_2)] = \sum_{U_3} (N_{U_1, U_2}^{U_3} - N_{U_2, U_1}^{U_3}) T_n(U_3)$$

□

**Remark 8.7.** *We may summarise these results by saying that the map from the Grothendieck ring of  $\mathcal{C}$  (with coefficients in  $\mathbb{Q}$ ) to the  $\mathbb{Q}$ -span of the  $T_n(U)$  defined by  $[U] \mapsto T_n(U)$  is a homomorphism of Lie algebras. The fact that  $T_n(M) = T_n(N) + T_n(M/N)$  shows “linearity”, and we have just shown that it preserves the Lie bracket.*

**8.2. Structure of the Limiting Grothendieck Ring.** We are now able to give a presentation of  $\mathcal{G}_{\infty}(\mathcal{C})$ . Recall that if  $A_i$  is an infinite family of unital algebras over  $k$ , then the tensor product  $\bigotimes_i A_i$  is spanned by pure tensors whose factors are the unit elements in their respective algebras for all but finitely many  $i$ .

**Theorem 8.8.** *The  $\mathbb{Q}$ -algebra structure on  $\mathcal{G}_\infty(\mathcal{C})$  is as follows:*

$$\mathcal{G}_\infty(\mathcal{C}) = \bigotimes_{i=1}^{\infty} U(\mathcal{G}(\mathcal{C})_i)$$

Here,  $U(\mathcal{G}(\mathcal{C})_i)$  is the universal enveloping algebra of the span of  $T_i(U)$  for  $U \in I(\mathcal{C})$  (this Lie algebra is contained within the  $i$ -th filtered component of  $\mathcal{G}_\infty$ ).

*Proof.* We have a map that takes  $[U] \in U(\mathcal{G}(\mathcal{C})_n)$  to  $T_n(U)$ . It is a homomorphism by Theorem 8.6. It is a bijection by Proposition 8.4; it is surjective because the  $T_n(U)$  generate  $\mathcal{G}_\infty(\mathcal{C})$  and it is injective because the map is an isomorphism upon taking associated graded algebras.  $\square$

**Remark 8.9.** *The previous theorem generalises the result that  $\text{Rep}(\mathcal{R} \wr S_t)$  is the free polynomial algebra generated by basic hooks when  $\mathcal{R}$  is cocommutative, as the universal enveloping algebra of an abelian Lie algebra is a free polynomial algebra.*

## 9. PARTITION COMBINATORICS

We now focus on finding an expression for  $X_\lambda$  in terms of the  $T_n(U)$ .

### 9.1. Irreducibles in Terms of $T_n(U)$ .

**Lemma 9.1.** *If  $D_{(n)}$  denotes the class function on  $S_n$  which is the indicator function of  $n$ -cycles, then the class function  $\text{Ind}_{S_n^m}(D_{(n)}^{\otimes m})$  is  $m!$  times the indicator function of cycle type  $(n^m)$ .*

*Proof.* This will clearly be some multiple of the indicator function of elements of cycle type  $(n^m)$ . The multiplicity can be found using the Frobenius character formula for induced representations, which demonstrates that it is in fact the index of  $S_n^m$  in its normaliser in  $S_{nm}$ . It is easy to check that the normaliser is the wreath product  $S_m \ltimes S_n^m$ , hence the index is  $m!$ .  $\square$

For now we will fix  $m$ , and consider relations between the  $T_m(U)$ .

**Definition 9.2.** *If  $\lambda$  is a partition of  $n$ , let;*

$$T_{m,\lambda}(a_1, a_2, \dots, a_n) = T_m(a_1 a_2 \dots a_{\lambda_1}) T_m(a_{\lambda_1+1} a_{\lambda_1+2} \dots a_{\lambda_1+\lambda_2}) \dots T_m(a_{n-\lambda_l(\lambda)+1} a_{n-\lambda_l(\lambda)+2} \dots a_n)$$

**Proposition 9.3.** *We have the following identity in  $\mathcal{G}_\infty(\mathcal{C})$ .*

$$T_m(a_1, a_2, \dots, a_n) = \sum_{\sigma \in S_n} \sum_{\lambda \vdash n} \frac{\varepsilon_\lambda}{z_\lambda} T_{m,\lambda}(a_{\sigma(1)} a_{\sigma(2)} \dots a_{\sigma(n)})$$

Before the proof, we note a useful corollary.

**Corollary 9.4.** *In the case where  $a_i = a$  for all  $i$ , writing  $\lambda = (1^{m_1} 2^{m_2} \dots n^{m_n})$  we obtain:*

$$T_m(a, a, \dots, a) = \sum_{\lambda \vdash n} \frac{\varepsilon_\lambda n!}{z_\lambda} T_m(a)^{m_1} T_m(a^2)^{m_2} \dots T_m(a^n)^{m_n}$$

*Proof.* We use Mackey theory to calculate  $T_m(b)T_m(a_1, a_2, \dots, a_n)$ . Note that the minimal length  $(S_m \times S_{(n-1)m+k}, S_m^n \times S_k)$ -double coset representatives either map the elements of  $\{1, 2, \dots, m\}$  to a contiguous block of  $m$  elements permuted by a single component group in  $S_m^n \times S_k$ , or the elements are split between such component groups. In the latter case, the corresponding terms (in the Mackey theory computation) will involve a nontrivial restriction to a Young subgroup, as before this will give zero. Thus, we consider the ways to pick a copy of  $S_m$  as one of the  $n$  given ones, or one contained in  $S_k$ . Analogously to before:

$$T_m(b)T_m(a_1, a_2, \dots, a_n) = T_m(b, a_1, a_2, \dots, a_n) + \sum_{i=1}^n T_m(a_1, \dots, a_{i-1}, b a_i, a_{i+1}, \dots, a_n)$$

Using this, we may decompose the claimed expression for  $T_m(a_1, a_2, \dots, a_n)$  into a linear combination of  $T_m(b_1, b_2, \dots, b_m)$ , where each  $b_j$  is a product of  $a_i$ s. We count the coefficient of a term of the following form in the expression on the right hand side of the claimed equality:

$$T_m(a_{r_{1,1}} a_{r_{1,2}} \dots a_{r_{1,q_1}}, \dots, a_{r_{k,1}} a_{r_{k,2}} \dots a_{r_{k,q_k}})$$

Here, the  $r_{i,j}$  for  $1 \leq i \leq k$ ,  $1 \leq j \leq q_i$  are exactly the numbers 1 through  $n$  in some order.

The argument  $a_{r_{1,1}} a_{r_{1,2}} \cdots a_{r_{1,q_1}}$  must arise from a product of terms such as the following:

$$T_m(a_{r_{1,1}} a_{r_{1,2}} \cdots a_{r_{1,\mu_1}}) T_m(a_{r_{1,\mu_1+1}} a_{r_{1,\mu_1+2}} \cdots a_{r_{1,\mu_1+\mu_2}}) \cdots T_m(a_{r_{1,q_1-\mu_l+1}} a_{r_{1,2}} \cdots a_{r_{1,q_k}})$$

Since all terms  $T_m(x)$  in the definition of  $T_{m,\lambda}$  occur in non-increasing order of the number of factors in the argument  $x$ , in this way we obtain a partition of  $q_k$ ,  $\mu^{(1)} = (\mu_1, \mu_2, \dots, \mu_l)$ , associated to the sequence  $r_{1,j}$  which describes the factors contributing to that term. A similar description holds for the other  $r_{i,j}$  for other values of  $i$ . In this way, we obtain a description of all contributions; note that the  $\lambda$  appearing in the sum will be the union of all  $\mu^{(i)}$ , and that if multiple  $\mu^{(i)}$  have parts of some size  $s$ , then there is no restriction on the ordering of the corresponding  $T_m(b_1 b_2 \cdots b_s)$  terms within  $T_{m,\lambda}$  (each possible ordering has an equal contribution). The coefficient of  $T_m(a_{r_{1,1}} a_{r_{1,2}} \cdots a_{r_{1,q_1}}, \dots, a_{r_{k,1}} a_{r_{k,2}} \cdots a_{r_{k,q_k}})$  is:

$$\sum_{\mu^{(1)} \vdash q_1} \sum_{\mu^{(2)} \vdash q_2} \cdots \sum_{\mu^{(k)} \vdash q_k} \frac{\varepsilon(\cup_i \mu^{(i)})}{z_{\cup_i \mu^{(i)}}} \prod_{j=1}^n \frac{(\sum_{i=1}^k m_j(\mu^{(i)}))!}{\prod_{i=1}^k m_j(\mu^{(i)})!}$$

Here the multinomial coefficient arose because different orderings of factors can give rise to the same term. Using the fact that  $\varepsilon(\mu \cup \nu) = \varepsilon(\mu)\varepsilon(\nu)$  and the definition of  $z_\mu$ , this becomes:

$$\sum_{\mu^{(1)} \vdash q_1} \sum_{\mu^{(2)} \vdash q_2} \cdots \sum_{\mu^{(k)} \vdash q_k} \prod_{i=1}^k \frac{\varepsilon(\mu^{(i)})}{z_{\mu^{(i)}}} = \prod_{i=1}^k \left( \sum_{\mu^{(i)} \vdash q_i} \frac{\varepsilon(\mu^{(i)})}{z_{\mu^{(i)}}} \right)$$

The expansions for elementary and complete symmetric functions in terms of power sum symmetric functions show that we have:

$$\delta_{n,1} = \langle s_{(n)}, s_{(1^n)} \rangle = \langle h_n, e_n \rangle = \sum_{\lambda \vdash n} \langle \frac{p_\lambda}{z_\lambda}, \frac{\varepsilon(\lambda) p_\lambda}{z_\lambda} \rangle = \sum_{\lambda \vdash n} \frac{\varepsilon(\lambda)}{z_\lambda}$$

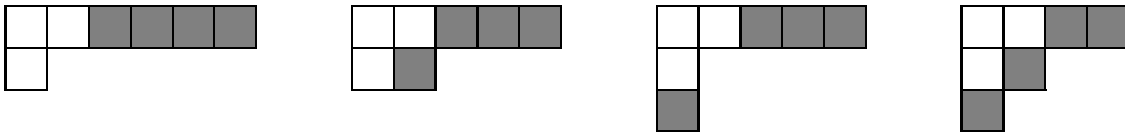
This means that the expression of interest vanishes unless each  $q_i = 1$ . In that case the constant is 1, and we simply obtain  $T(a_1, a_2, \dots, a_n)$  as claimed.  $\square$

**Proposition 9.5.** *We may relate  $U^{\boxtimes |\lambda|} \otimes S^\lambda$  to the  $T_i(U)$  as follows.*

$$\lim_{m \rightarrow \infty} \text{Ind}_{S_{|\lambda|} \times S_m}^{S_{|\lambda|+m}} \left( (U^{\boxtimes |\lambda|} \otimes S^\lambda) \boxtimes (\mathbf{1}^{\boxtimes m} \otimes \mathbf{1}_{S_m}) \right) = \sum_{\mu \vdash |\lambda|} \chi_\mu^\lambda \frac{T_1(\overbrace{U, U, \dots, U}^{m_1(\mu)})}{m_1(\mu)!} \frac{T_2(\overbrace{U, U, \dots, U}^{m_2(\mu)})}{m_2(\mu)!} \cdots \frac{T_{|\lambda|}(\overbrace{U, U, \dots, U}^{m_{|\lambda|}(\mu)})}{m_{|\lambda|}(\mu)!}$$

*Proof.* We decompose  $S^\lambda$  into a linear combination of virtual representations, each having character equal to an indicator function of some cycle type  $\mu$ . The scalar multiple will clearly be  $\chi_\mu^\lambda$ . By Lemma 9.1,  $\frac{T_i(U, U, \dots, U)}{m_i(\mu)!}$  will correspond to the indicator function of cycle type  $(i^{m_i(\mu)})$ . Multiplying these together corresponds to taking the tensor product of class functions on  $S_{i^{m_i(\mu)}}$  and inducing up to  $S_n$ , which precisely gives the indicator function of cycle type  $\mu$ .  $\square$

**Remark 9.6.** *Suppose that we wish to decompose the expression in Proposition 9.5 into  $X_\mu$ . It is clear that if  $U \neq \mathbf{1}$  then we get the definition of  $X_\mu$  where  $\mu(U) = \lambda$  and  $\mu(V)$  is the trivial partition for  $V \neq U$ . If  $U = \mathbf{1}$ , we use Proposition 4.5 to see that we must describe  $\text{Ind}_{S_{|\lambda|} \times S_m}^{S_{|\lambda|+m}} (S^\lambda \boxtimes \mathbf{1}_{S_m})$  for  $m$  sufficiently large. Under (the inverse of) the characteristic map between symmetric functions and representations of symmetric groups, this amounts to calculating the product  $s_\lambda h_m$ , which is described combinatorially by the Pieri rule which states that we get  $\sum_\mu s_\mu$  where the sum across all partitions  $\mu$  obtained from  $\lambda$  by adding  $m$  boxes, no two in the same column. For example, suppose  $\lambda = (2, 1)$  and  $m = 4$ . The valid  $\mu$  are shown below, where the added boxes are highlighted.*





When  $m$  is larger than the number of columns (i.e. the longest part of  $\lambda$ ), there is no restriction on the collection of columns that a box may be added to, save that the final result must be a partition. We are interested in the set of partitions obtained by removing the first row of each of the diagrams after performing the above operation. That operation is the same as removing a box from each column. Thus, we are interested in all partitions obtained by adding at most one box to some column, and then removing one box from each column. This is equivalent to removing one box from each of the columns in the diagram of  $\lambda$  that were not chosen. In other words, the set we are interested in consists of all partitions obtained from  $\lambda$  by removing some number of boxes, no two in the same column. If we write  $h_r^\perp$  for the operator adjoint to multiplication by  $h_r$  with respect to the usual bilinear form on  $\Lambda$ , then it is clear that  $h_r^\perp s_\lambda$  is  $\sum_\mu s_\mu$  across all partitions  $\mu$  obtained from  $\lambda$  by removing  $r$  boxes in the diagram of  $\lambda$ , no two in the same column. Continuing to encode partitions as their associated Schur functions, we find that the desired decomposition is

$$\left( \sum_{r=0}^{\infty} h_r^\perp \right) s_\lambda$$

This is because, for  $m$  sufficiently large, there is no restriction on the number of boxes that could be removed.

**Example 9.7.** Suppose that in Proposition 9.5,  $\lambda = (1^r)$  and  $U = \mathbf{1}$ . Since  $h_i^\perp s_{(1^r)} = 0$  for  $i \geq 2$ , and  $h_1^\perp s_{(1^r)} = s_{(1^{r-1})}$  (and  $h_0^\perp s_{(1^r)} = s_{(1^r)}$ ), we obtain

$$\lim_{m \rightarrow \infty} \text{Ind}_{S_r \times S_m}^{S_{r+m}} \left( \left( \mathbf{1}^{\boxtimes r} \otimes S^{(1^r)} \right) \boxtimes \left( \mathbf{1}^{\boxtimes m} \otimes \mathbf{1}_{S_m} \right) \right) = X_{\mu_1} + X_{\mu_2}$$

where  $\mu_1(\mathbf{1}) = (1^r)$  and  $\mu_2(\mathbf{1}) = (1^{r-1})$  (and all parts of these multipartitions associated to  $U \neq \mathbf{1}$  are trivial).

**Remark 9.8.** To reconstruct  $X_\lambda$  from the objects in Proposition 9.5, we need to invert the operator  $\sum_{r=0}^{\infty} h_r^\perp$ . Recognising it as the adjoint of  $H(1)$ , we may write the inverse as the adjoint of  $E(-1)$  (recall that  $H(t)E(-t) = 1$ ). So, the relevant operator (when we are encoding partitions as Schur functions) is  $\sum_{r=0}^{\infty} (-1)^r e_r^\perp$  (where  $e_r^\perp$  is adjoint to multiplication by  $e_r$ ).

**Proposition 9.9.** Let  $U$  be an object of  $\mathcal{C}$ , and  $\varphi_U : \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \mathcal{G}_{\infty}(\mathcal{C})$  be defined by

$$\varphi_U(e_i) = \lim_{m \rightarrow \infty} \text{Ind}_{S_i \times S_m}^{S_{i+m}} \left( \left( U^{\boxtimes i} \otimes S^{(1^i)} \right) \boxtimes \left( \mathbf{1}^{\boxtimes m} \otimes \mathbf{1}_{S_m} \right) \right)$$

so that if  $U \neq \mathbf{1}$  is a simple object, then  $\varphi_U(e_i)$  is the basic hook  $e_i(U)$ , whilst Example 9.7 shows that if  $U = \mathbf{1}$ , then  $\varphi(e_i) = e_i(\mathbf{1}) + e_{i-1}(\mathbf{1})$ . Then:

$$\varphi_U(p_n) = \sum_{d|n} d T_d(U^{\frac{n}{d}})$$

*Proof.* We use the generating functions  $E(t) = \sum_{n=0}^{\infty} e_n t^n$ , and  $P(t) = \sum_{n=0}^{\infty} p_{n+1} t^n$ . Recall that

$$\frac{d}{dt} \log(E(t)) = P(-t)$$

Additionally, we have the following expression using Proposition 9.5, following directly from the character formula for the sign representation.

$$\begin{aligned} \varphi_U(e_n) &= \sum_{\lambda \vdash n} \varepsilon_\lambda \frac{T_1(\overbrace{U, U, \dots, U}^{m_1(\lambda)})}{m_1(\lambda)!} \frac{T_2(\overbrace{U, U, \dots, U}^{m_2(\lambda)})}{m_2(\lambda)!} \dots \frac{T_n(\overbrace{U, U, \dots, U}^{m_n(\lambda)})}{m_n(\lambda)!} \\ &= \sum_{\lambda \vdash n} \prod_{i=1}^n \frac{(-1)^{m_i(\lambda)(i-1)}}{m_i(\lambda)!} T_i(\overbrace{U, U, \dots, U}^{m_i(\lambda)}) \\ &= \sum_{\lambda \vdash n} \prod_{i=1}^n (-1)^{m_i(\lambda)(i-1)} \sum_{\mu^{(i)} \vdash m_i(\lambda)} \frac{\varepsilon_{\mu^{(i)}}}{z_{\mu^{(i)}}} \prod_j T_i(U^j)^{m_j(\mu^{(i)})} \end{aligned}$$

In the last step, we used Corollary 9.4. We now calculate the generating function  $E(t)$ .

$$\begin{aligned}
\varphi_U(E(t)) &= \sum_{\lambda \vdash n} t^{|\lambda|} \prod_{i=1}^n (-1)^{m_i(\lambda)(i-1)} \sum_{\mu^{(i)} \vdash m_i(\lambda)} \frac{\varepsilon_{\mu^{(i)}}}{z_{\mu^{(i)}}} \prod_j T_i(U^j)^{m_j(\mu^{(i)})} \\
&= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \dots \left( \prod_{i=1}^{\infty} t^{im_i} (-1)^{(i-1)m_i} \sum_{\mu^{(i)} \vdash m_i} \frac{\varepsilon_{\mu^{(i)}}}{z_{\mu^{(i)}}} \prod_j T_i(U^j)^{m_j(\mu^{(i)})} \right) \\
&= \prod_{i=1}^{\infty} \sum_{m_i=0}^{\infty} \left( t^{im_i} (-1)^{(i-1)m_i} \sum_{\mu^{(i)} \vdash m_i} \frac{\varepsilon_{\mu^{(i)}}}{z_{\mu^{(i)}}} \prod_j T_i(U^j)^{m_j(\mu^{(i)})} \right) \\
&= \prod_{i=1}^{\infty} \sum_{m_i=0}^{\infty} \left( t^{im_i} (-1)^{(i-1)m_i} \sum_{\mu^{(i)} \vdash m_i} \prod_j \frac{T_i(U^j)^{m_j(\mu^{(i)})} (-1)^{m_j(\mu^{(i)})(j-1)}}{m_j(\mu^{(i)})! j^{m_j(\mu^{(i)})}} \right) \\
&= \prod_{i=1}^{\infty} \left( \sum_{m_i=0}^{\infty} \sum_{\mu^{(i)} \vdash m_i} \prod_{j=1}^{\infty} t^{ijm_j(\mu^{(i)})} (-1)^{(i-1)jm_j(\mu^{(i)})} \frac{T_i(U^j)^{m_j(\mu^{(i)})} (-1)^{m_j(\mu^{(i)})(j-1)}}{m_j(\mu^{(i)})! j^{m_j(\mu^{(i)})}} \right) \\
&= \prod_{i=1}^{\infty} \left( \sum_{m_i=0}^{\infty} \sum_{\mu^{(i)} \vdash m_i} \prod_{j=1}^{\infty} \frac{1}{m_j(\mu^{(i)})!} \left( t^{ij} (-1)^{(ij-1)} \frac{T_i(U^j)}{j} \right)^{m_j(\mu^{(i)})} \right) \\
&= \prod_{i=1}^{\infty} \left( \sum_{\mu^{(i)} \in \mathcal{P}} \prod_{j=1}^{\infty} \frac{1}{m_j(\mu^{(i)})!} \left( t^{ij} (-1)^{(ij-1)} \frac{T_i(U^j)}{j} \right)^{m_j(\mu^{(i)})} \right) \\
&= \prod_{i=1}^{\infty} \left( \exp \left( \sum_{j=1}^{\infty} t^{ij} (-1)^{(ij-1)} \frac{T_i(U^j)}{j} \right) \right)
\end{aligned}$$

In the second last step, we used the fact that the set of integer partitions  $\mu$  is parametrised by the numbers  $m_j(\mu) \in \mathbb{Z}_{\geq 0}$  with all but finitely many being zero. This allows us to sum over the numbers  $m_j(\mu^{(i)})$  independently of each other. Now we may take the derivative of the logarithm with respect to  $t$ :

$$\begin{aligned}
\varphi_U \left( \frac{E'(t)}{E(t)} \right) &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} i j t^{ij-1} (-1)^{(ij-1)} \frac{T_i(U^j)}{j} \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} i (-t)^{ij-1} T_i(U^j) \\
&= \sum_{n=1}^{\infty} \sum_{d|n} d (-t)^{n-1} T_d(U^{n/d})
\end{aligned}$$

In the last step, the change of variables  $i = d$ ,  $ij = n$  was used. Equating the terms of the power series with those of  $\varphi_U(P(-t))$  gives the result.  $\square$

**9.2. Generating Function for Irreducibles.** We now prove the main theorem of this paper.

**Theorem 9.10.** Write  $\Lambda_{\mathbb{Q}}^{(U)}$  for a copy of the ring of symmetric functions with rational coefficients, whose variables we associate with  $U \in I(\mathcal{C})$ . If  $f$  is a symmetric function, we write  $f^{(U)}$  to denote  $f$  considered as an element of  $\Lambda_{\mathbb{Q}}^{(U)}$ . Define a bilinear form  $\langle -, - \rangle : \mathcal{G}(\mathcal{C}) \times \mathcal{G}(\mathcal{C}) \rightarrow \mathbb{Z}$  by  $\langle [U'], [U] \rangle = \delta_{U, U'}$ . We have the following equality of formal series in  $\left( \bigotimes_{U \in I(\mathcal{C})} \Lambda_{\mathbb{Q}}^{(U)} \right) \hat{\otimes} \mathcal{G}_{\infty}$  (where we have used the completed tensor product):

$$\sum_{\lambda \in \mathcal{P}_{\mathcal{C}}} \left( \prod_{U \in I(\mathcal{C})} s_{\lambda(U)}^{(U)} \right) \otimes X_{\lambda} = \left( \sum_{r \geq 0} (-1)^r e_r^{(1)} \right) \prod_{l=1}^{\infty} \exp \left( \sum_{U' \in I(\mathcal{C})} \left\langle [U'], \log \left( 1 + \sum_U p_l^{(U)} [U] \right) \right\rangle \otimes T_l(U') \right)$$

*Proof.* We firstly note that the first factor on the right hand side can be inverted (using the generating function relation  $H(t)E(-t) = 1$ ). Moving it to the left hand side it acts as an operator on the symmetric functions, but by taking the adjoint, we may make it act of the  $X_\lambda$ . Taking into consideration Remark 9.8, we see that the effect of this is to replace  $X_\lambda$  with  $\lim_{m \rightarrow \infty} \text{Ind}_{S_\lambda \times S_m}^{S_{|\lambda|+m}} (\boxtimes_{U \in I(\mathcal{C})} (U^{\boxtimes|\lambda|} \otimes S^{\lambda(U)}) \boxtimes (\mathbf{1}^{\boxtimes m} \otimes \mathbf{1}_{S_m}))$ , for which Proposition 9.5 gives an expression. The left hand side of the equation becomes

$$\sum_{\lambda \in \mathcal{P}_\mathcal{C}} \left( \prod_{U \in I(\mathcal{C})} s_{\lambda(U)}^{(U)} \right) \otimes \left( \lim_{m \rightarrow \infty} \text{Ind}_{S_\lambda \times S_m}^{S_{|\lambda|+m}} (\boxtimes_{U \in I(\mathcal{C})} (U^{\boxtimes|\lambda(U)|} \otimes S^{\lambda(U)}) \boxtimes (\mathbf{1}^{\boxtimes m} \otimes \mathbf{1}_{S_m})) \right)$$

and we are required to prove that it is equal to

$$\prod_{s=1}^{\infty} \exp \left( \sum_{U' \in I(\mathcal{C})} \left\langle U', \log \left( 1 + \sum_U p_s^{(U)} U \right) \right\rangle \otimes T_s(U') \right)$$

Using the same method as in the proof of Proposition 9.5, we seek to write the expression in terms of the elements  $T_i(U)$ . We have

$$\begin{aligned} & \lim_{m \rightarrow \infty} \text{Ind}_{S_\lambda \times S_m}^{S_{|\lambda|+m}} (\boxtimes_{U \in I(\mathcal{C})} (U^{\boxtimes|\lambda(U)|} \otimes S^{\lambda(U)}) \boxtimes (\mathbf{1}^{\boxtimes m} \otimes \mathbf{1}_{S_m})) \\ &= \sum_{\mu^{(1)}} \sum_{\mu^{(2)}} \cdots \left( \prod_{U_i \in I(\mathcal{C})} \chi_{\mu^{(i)}}^{\lambda(U_i)} \right) \prod_{i=1}^{\infty} \frac{T_i(\overbrace{U_1, U_1, \dots}^{m_i(\mu^{(1)})}, \overbrace{U_2, U_2, \dots}^{m_i(\mu^{(2)})}, \dots)}{m_1(\mu^{(1)})! m_1(\mu^{(2)})! \dots} \end{aligned}$$

Here  $\mu^{(i)} \in \mathcal{P}$  describes a cycle type in a symmetric group associated to  $U_i \in I(\mathcal{C})$ . We now let  $\nu = \cup_i \mu^{(i)}$  and use Proposition 9.3 to express this in terms of  $T_i(U)$  (i.e. without any inductions). We obtain

$$\begin{aligned} & \sum_{\mu^{(1)}} \sum_{\mu^{(2)}} \cdots \left( \prod_{U_i \in I(\mathcal{C})} \chi_{\mu^{(i)}}^{\lambda(U_i)} \right) \left( \prod_{j=1}^{\infty} \frac{1}{m_j(\mu^{(1)})! m_j(\mu^{(2)})! \dots} \right) \\ & \times \sum_{\alpha^{(1)} \vdash m_1(\nu)} \sum_{\alpha^{(2)} \vdash m_2(\nu)} \cdots \left( \frac{\varepsilon_{\alpha^{(1)}}}{z_{\alpha^{(1)}}} \frac{\varepsilon_{\alpha^{(2)}}}{z_{\alpha^{(2)}}} \dots \right) \prod_{l=1}^{\infty} \sum_{\sigma \in S_{m_l(\nu)}} T_{l, \alpha^{(l)}}(\sigma(\overbrace{U_1, U_1, \dots}^{m_l(\mu^{(1)})}, \overbrace{U_2, U_2, \dots}^{m_l(\mu^{(2)})}, \dots)) \end{aligned}$$

Ultimately we are calculating a generating function whose inner product with the symmetric function  $\prod_{U \in I(\mathcal{C})} s_{\lambda(U)}^{(U)}$  is the above quantity. If  $\sigma$  and  $\rho$  are partitions, then the fact that the inner product of  $s_\sigma$  and  $p_\rho$  is  $\chi_\rho^\sigma$  allows us to replace the character values with power-sum symmetric functions and sum over all possible values of these. We also note that the symmetric group action of  $S_{m_i(\nu)}$  is trivial when restricted to  $S_{m_i(\mu^{(1)})} \times S_{m_i(\mu^{(2)})} \times \dots$ . This means that we may restrict the sum to coset representatives of this subgroup at the cost of multiplying by  $m_i(\mu^{(1)})! m_i(\mu^{(2)})! \dots$  (which cancels out the denominators following the symmetric group characters in the above expression).

$$\begin{aligned} & \sum_{\mu^{(1)}} \sum_{\mu^{(2)}} \cdots \left( \prod_{U_i \in I(\mathcal{C})} p_{\mu^{(i)}}^{(U_i)} \right) \otimes \sum_{\alpha^{(1)} \vdash m_1(\nu)} \sum_{\alpha^{(2)} \vdash m_2(\nu)} \cdots \left( \frac{\varepsilon_{\alpha^{(1)}}}{z_{\alpha^{(1)}}} \frac{\varepsilon_{\alpha^{(2)}}}{z_{\alpha^{(2)}}} \dots \right) \\ & \times \prod_{l=1}^{\infty} \sum_{\sigma \in S_{m_l(\nu)} / S_{m_l(\mu^{(1)})} \times S_{m_l(\mu^{(2)})} \dots} T_{l, \alpha^{(l)}}(\sigma(\overbrace{U_1, U_1, \dots}^{m_l(\mu^{(1)})}, \overbrace{U_2, U_2, \dots}^{m_l(\mu^{(2)})}, \dots)) \end{aligned}$$

Now, note that the  $T_{\alpha^{(i)}}(\dots)$  are summed over all distinct reorderings of their arguments. We now inspect the sum over  $\alpha^{(l)}$  and  $\sigma$  more closely (which we take to include the terms  $(p_l^{(U_1)})_{m_l(\mu^{(1)})}(p_l^{(U_2)})_{m_l(\mu^{(2)})} \dots$ ).

$$\begin{aligned}
& (p_l^{(U_1)})_{m_l(\mu^{(1)})}(p_l^{(U_2)})_{m_l(\mu^{(2)})} \dots \otimes \sum_{\alpha^{(l)} \vdash m_l(\nu)} \frac{\varepsilon_{\alpha^{(l)}}}{z_{\alpha^{(l)}}} \\
& \times \sum_{\sigma \in S_{m_l(\nu)} / S_{m_l(\mu^{(1)})} \times S_{m_l(\mu^{(2)})} \dots} T_{l, \alpha^{(l)}}(\sigma(\overbrace{U_1, U_1, \dots}^{m_l(\mu^{(1)})}, \overbrace{U_2, U_2, \dots}^{m_l(\mu^{(2)})}, \dots))
\end{aligned}$$

Recalling that  $\frac{\varepsilon_{\alpha^{(l)}}}{z_{\alpha^{(l)}}} = \prod_{j=1}^{\infty} \frac{(-1)^{m_j(\alpha^{(l)})} (j-1)}{m_j(\alpha^{(l)})! j^{m_j(\alpha^{(l)})}}$ , we have

$$\begin{aligned}
& \left( \prod_{U_i \in I(\mathcal{C})} (p_l^{(U_i)})_{m_l(\mu^{(i)})} \right) \otimes \sum_{\alpha^{(l)} \vdash m_l(\nu)} \prod_{j=1}^{\infty} \frac{(-1)^{m_j(\alpha^{(l)})} (j-1)}{m_j(\alpha^{(l)})! j^{m_j(\alpha^{(l)})}} \\
& \times \sum_{\sigma \in S_{m_l(\nu)} / S_{m_l(\mu^{(1)})} \times S_{m_l(\mu^{(2)})} \dots} T_{l, \alpha^{(l)}}(\sigma(\overbrace{U_1, U_1, \dots}^{m_l(\mu^{(1)})}, \overbrace{U_2, U_2, \dots}^{m_l(\mu^{(2)})}, \dots))
\end{aligned}$$

We sum over all possible values of  $m_l(\mu^{(1)}), m_l(\mu^{(2)}), \dots$ , which means that  $\sigma(\overbrace{U_1, U_1, \dots}^{m_l(\mu^{(1)})}, \overbrace{U_2, U_2, \dots}^{m_l(\mu^{(2)})}, \dots)$  varies across all finite words  $W$  in the  $U_i$  without repetition. To calculate  $T_{l, \alpha^{(l)}}(W)$  we write  $W_{\alpha^{(l)}, r}$  for the product of the letters of the subword of  $W$  starting at the  $(\alpha_1^{(l)} + \alpha_2^{(l)} + \dots + \alpha_{r-1}^{(l)} + 1)$ -th place and finishing at the  $(\alpha_1^{(l)} + \alpha_2^{(l)} + \dots + \alpha_r^{(l)})$ -th place. This lets us write

$$T_{l, \alpha^{(l)}} = T_l(W_{\alpha^{(l)}, 1}) T_l(W_{\alpha^{(l)}, 2}) \dots T_l(W_{\alpha^{(l)}, l(\alpha^{(l)})})$$

Now, in  $\mathcal{G}(\mathcal{C})$  we may write  $[W_{\alpha^{(l)}, r}] = \sum_{U \in I(\mathcal{C})} M_{W, \alpha^{(l)}, r}^U [U]$ , and because  $T_l(-)$  is linear,

$$T_l(W_{\alpha^{(l)}, r}) = \sum_{U \in I(\mathcal{C})} M_{W, \alpha^{(l)}, r}^{(U)} T_l(U) = \sum_{U \in I(\mathcal{C})} \langle [U], [W_{\alpha^{(l)}, r}] \rangle T_l(U)$$

If we write  $|W|$  for the length of the word  $W$ , and  $n_U(W)$  for the number of occurrences of  $U$  in  $W$ , then we may rewrite our earlier expression as

$$\begin{aligned}
& \sum_W \left( \prod_{U_i \in I(\mathcal{C})} (p_l^{(U_i)})_{n_U(W)} \right) \otimes \left( \sum_{\alpha^{(l)} \vdash |W|} \prod_{j=1}^{\infty} \frac{(-1)^{m_j(\alpha^{(l)})} (j-1)}{m_j(\alpha^{(l)})! j^{m_j(\alpha^{(l)})}} \right) \left( \prod_r T_l(W_{\alpha^{(l)}, r}) \right) \\
& = \sum_W \left( \prod_{U_i \in I(\mathcal{C})} (p_l^{(U_i)})_{n_U(W)} \right) \otimes \left( \sum_{\alpha^{(l)} \vdash |W|} \prod_{j=1}^{\infty} \frac{(-1)^{m_j(\alpha^{(l)})} (j-1)}{m_j(\alpha^{(l)})! j^{m_j(\alpha^{(l)})}} \right) \left( \prod_r \sum_{U \in I(\mathcal{C})} \langle [U], [W_{\alpha^{(l)}, r}] \rangle T_l(U) \right) \\
& = \sum_W \left( \sum_{\alpha^{(l)} \vdash |W|} \prod_{j=1}^{\infty} \frac{(-1)^{m_j(\alpha^{(l)})} (j-1)}{m_j(\alpha^{(l)})! j^{m_j(\alpha^{(l)})}} \right) \left( \prod_r \sum_{U \in I(\mathcal{C})} \langle [U], [W_{\alpha^{(l)}, r}] \rangle \left( \prod_{U_i \in I(\mathcal{C})} (p_l^{(U_i)})_{n_U(W_{\alpha^{(l)}, r})} \right) \otimes T_l(U) \right)
\end{aligned}$$

We now note that each  $W_{\alpha^{(l)}, r}$  varies independently over all words in the  $U_i$  of length  $\alpha_r^{(l)}$ . We may therefore remove the sum over  $W$  at the cost of replacing

$$\langle [U], [W_{\alpha^{(l)}, r}] \rangle \left( \prod_{U_i \in I(\mathcal{C})} (p_l^{(U_i)})_{n_U(W_{\alpha^{(l)}, r})} \right)$$

with

$$\left\langle [U], \left( \sum_{U' \in I(\mathcal{C})} p_l^{U'} [U'] \right)^{\alpha_r^{(l)}} \right\rangle$$

This leaves us with

$$\begin{aligned}
& \sum_{\alpha^{(l)} \in \mathcal{P}} \left( \prod_{j=1}^{\infty} \frac{(-1)^{m_j(\alpha^{(l)})(j-1)}}{m_j(\alpha^{(l)})! j^{m_j(\alpha^{(l)})}} \right) \left( \prod_r \sum_{U \in I(\mathcal{C})} \left\langle [U], \left( \sum_{U' \in I(\mathcal{C})} p_l^{U'}[U'] \right)^{\alpha_r^{(l)}} \right\rangle \otimes T_l(U) \right) \\
&= \sum_{\alpha^{(l)} \in \mathcal{P}} \prod_{j=1}^{\infty} \frac{1}{m_j(\alpha^{(l)})!} \left( \frac{(-1)^{(j-1)}}{j} \right)^{m_j(\alpha^{(l)})} \left( \sum_{U \in I(\mathcal{C})} \left\langle [U], \left( \sum_{U' \in I(\mathcal{C})} p_l^{U'}[U'] \right)^j \right\rangle \otimes T_l(U) \right)^{m_j(\alpha^{(l)})} \\
&= \prod_{j=1}^{\infty} \sum_{m_j(\alpha^{(l)})=0}^{\infty} \frac{1}{m_j(\alpha^{(l)})!} \left( \frac{(-1)^{(j-1)}}{j} \right)^{m_j(\alpha^{(l)})} \left( \sum_{U \in I(\mathcal{C})} \left\langle [U], \left( \sum_{U' \in I(\mathcal{C})} p_l^{U'}[U'] \right)^j \right\rangle \otimes T_l(U) \right)^{m_j(\alpha^{(l)})}
\end{aligned}$$

Here we used the fact that summing over all partitions  $\alpha^{(l)}$  is equivalent to summing over all possible values of  $m_r(\alpha^{(l)})$  for all  $r$ . Now we recognise the power series for the exponential and then for the logarithm.

$$\begin{aligned}
& \prod_{j=1}^{\infty} \exp \left( \sum_{U \in I(\mathcal{C})} \left\langle [U], \frac{(-1)^{(j-1)}}{j} \left( \sum_{U' \in I(\mathcal{C})} p_l^{U'}[U'] \right)^j \right\rangle \otimes T_l(U) \right) \\
&= \exp \left( \sum_{U \in I(\mathcal{C})} \left\langle [U], \sum_{j=1}^{\infty} \frac{(-1)^{(j-1)}}{j} \left( \sum_{U' \in I(\mathcal{C})} p_l^{U'}[U'] \right)^j \right\rangle \otimes T_l(U) \right) \\
&= \exp \left( \sum_{U \in I(\mathcal{C})} \left\langle [U], \log \left( 1 + \sum_{U' \in I(\mathcal{C})} p_l^{U'}[U'] \right) \right\rangle \otimes T_l(U) \right)
\end{aligned}$$

Now we simply multiply this expression for  $l \in \mathbb{Z}_{>0}$  to obtain the desired result (since  $T_{l_1}(U)$  commutes with  $T_{l_2}(V)$  whenever  $l_1 \neq l_2$ , we do not need to be careful about commuting exponentials).  $\square$

In order to obtain expressions for  $X_\lambda$  in terms of basic hooks, we must write  $T_s(U)$  in terms of basic hooks.

**Proposition 9.11.** *Recall the setting of Proposition 9.9, where for any object  $U$  of  $\mathcal{C}$ , we had*

$$\varphi_U(p_n) = \sum_{d|n} d T_d(U^{\frac{n}{d}})$$

We have:

$$T_r(U) = \frac{1}{r} \sum_{d|r} \varphi_{(U^{\frac{r}{d}})}(p_d) \mu(r/d)$$

*Proof.* We directly calculate

$$\begin{aligned}
\frac{1}{r} \sum_{d|r} \varphi_{(U^{\frac{r}{d}})}(p_d) \mu(r/d) &= \frac{1}{r} \sum_{d|r} \mu(r/d) \left( \sum_{d'|d} d' T_{d'}((U^{\frac{r}{d}})^{\frac{d}{d'}}) \right) \\
&= \frac{1}{r} \sum_{d|r} \mu(r/d) \left( \sum_{d'|d} d' T_{d'}(U^{\frac{r}{d'}}) \right) \\
&= \frac{1}{r} \sum_{d'|r} \left( \sum_{d'|d|r} \mu(r/d) \right) d' T_{d'}(U^{\frac{r}{d'}}) \\
&= \frac{1}{r} \sum_{d'|r} \delta_{d',r} d' T_{d'}(U^{\frac{r}{d'}}) \\
&= T_r(U)
\end{aligned}$$

□

This means that to express the  $T_r(U)$  in terms of basic hooks, it is enough to decompose  $\varphi_{(U^{\frac{r}{d}})}(p_d)$  into basic hooks. This task is complicated by the fact that  $\varphi_{(V)}(p_d)$  is not linear in  $V$  for  $d > 1$ . However, this difficulty is mitigated if  $U^{\frac{r}{d}}$  is itself a simple object of  $\mathcal{C}$ . One case when this happens is when  $\mathcal{C} = G - \text{mod}$  where  $G$  is an abelian group (simple objects are precisely one dimensional representations of  $G$ , the set of which is closed under taking tensor products). In this case the problem amounts to expressing power sum symmetric functions in terms of elementary symmetric functions. The elementary symmetric functions give rise to basic hooks if for simple  $U \neq \mathbf{1}$ , and to a sum of two basic hooks for  $U = \mathbf{1}$ , as per Proposition 9.9.

## 10. APPLICATIONS TO SYMMETRIC GROUPS AND WREATH PRODUCTS

We discuss a selection of results about the asymptotic representation theory of symmetric groups and wreath products that follow from our results. Recall that the Deligne category  $\underline{\text{Rep}}(S_t)$  is a tensor category that can be thought of as an “interpolation” of the representation categories of finite symmetric groups.

**Theorem 10.1.** *The ring  $\mathcal{G}_\infty(\mathcal{C})$  is isomorphic to the Grothendieck ring (with rational coefficients) of the wreath product version of the Deligne category,  $\underline{\text{Rep}}(\mathcal{R} \wr S_t)$ , when  $t \notin \mathbb{Z}_{\geq 0}$ . The Grothendieck ring with integral coefficients is isomorphic to the integral version of  $\mathcal{G}_\infty(\mathcal{C})$  described in Remark 7.10.*

*Proof.* When  $t \notin \mathbb{Z}_{\geq 0}$ , the simple objects of the category  $\text{Rep}(\mathcal{R} \wr S_t)$  are parametrised by  $\lambda \in \mathcal{P}_{\mathcal{C}}$ . In this situation, the methods of Theorem 7.12 allow one to deduce that the structure constants for non-integral  $t$  agree with the corresponding stable limits as  $t \in \mathbb{Z}_{\geq 0}$  tends to infinity. □

The wreath product categories are discussed in [Mor12], and various aspects of the theory of Deligne categories are discussed in [Eti14] and [Eti16]. We now give a way for computing a formula for structure constants of  $\mathcal{G}_\infty(\mathcal{C})$  with respect to the  $X_\lambda$  basis in the case where  $\mathcal{G}(\mathcal{C})$  is commutative (in particular, for  $\mathcal{C} = kG - \text{mod}$  for a group  $G$ ). Of course, these are also the structure constants in the Grothendieck ring of a Deligne category. We use Theorem 9.10 with multiple different sets of symmetric function variables. It will be convenient to write  $p_l(\underline{x}^{(U)})$  instead of  $p_l^{(U)}(\underline{x})$ .

**Theorem 10.2.** *Suppose that  $\mathcal{G}(\mathcal{C})$  is commutative and write  $N_{U,V}^W$  for the structure tensor (so that  $[U][V] = \sum_W N_{U,V}^W [W]$ ). Write  $\underline{z}^{(U)}$  to denote the family of symmetric function variables  $\bigoplus_{V_1, V_2 \in I(\mathcal{C})} (\underline{x}^{(V_1)} \underline{y}^{(V_2)})^{\oplus N_{V_1, V_2}^U}$ , where direct sum notation denotes a collection of symmetric function variables, and the direct sum in the exponents denotes the multiplicity of each of the sets of variables. Then the multiplicity of  $X_\lambda$  in  $X_{\lambda_1} X_{\lambda_2}$  is given by the coefficient of*

$$\left( \prod_{U \in I(\mathcal{C})} s_{\lambda_1(U)}(\underline{x}^{(U)}) \prod_{V \in I(\mathcal{C})} s_{\lambda_2(V)}(\underline{y}^{(V)}) \right)$$

in

$$\prod_{U, V \in I(\mathcal{C})} \left( \sum_{\rho \in \mathcal{P}} s_\rho(\underline{x}^{(U)}) s_\rho(\underline{y}^{(V)}) \right)^{N_{U,V}^{(1)}} \sum_{\lambda \in \mathcal{P}_{\mathcal{C}}} \left( \prod_{U \in I(\mathcal{C})} s_{\lambda(U)}(\underline{x}^{(U)}, \underline{y}^{(U)}, \underline{z}^{(U)}) \right) \otimes X_\lambda$$

*Proof.* We manipulate generating functions, starting with one for which the theorem would clearly hold.

$$\begin{aligned}
& \sum_{\lambda_1 \in \mathcal{P}_c} \sum_{\lambda_2 \in \mathcal{P}_c} \left( \prod_{U \in I(\mathcal{C})} s_{\lambda_1(U)}(\underline{x}^{(U)}) \prod_{V \in I(\mathcal{C})} s_{\lambda_2(V)}(\underline{y}^{(V)}) \right) \otimes (X_{\lambda_1} X_{\lambda_2}) \\
&= \left( \sum_{\lambda_1 \in \mathcal{P}_c} \left( \prod_{U \in I(\mathcal{C})} s_{\lambda_1(U)}(\underline{x}^{(U)}) \right) \otimes X_{\lambda_1} \right) \left( \sum_{\lambda_2 \in \mathcal{P}_c} \left( \prod_{V \in I(\mathcal{C})} s_{\lambda_2(V)}(\underline{y}^{(V)}) \right) \otimes X_{\lambda_2} \right) \\
&= \left( \sum_{r \geq 0} (-1)^r e_r(\underline{x}^{(1)}) \right) \prod_{l=1}^{\infty} \exp \left( \sum_{U' \in I(\mathcal{C})} \left\langle [U'], \log \left( 1 + \sum_U p_l(\underline{x}^{(U)})[U] \right) \right\rangle \otimes T_l(U') \right) \\
&\times \left( \sum_{r \geq 0} (-1)^r e_r(\underline{y}^{(1)}) \right) \prod_{l=1}^{\infty} \exp \left( \sum_{V' \in I(\mathcal{C})} \left\langle [V'], \log \left( 1 + \sum_V p_l(\underline{y}^{(V)})[V] \right) \right\rangle \otimes T_l(V') \right)
\end{aligned}$$

Since we assumed  $\mathcal{G}(\mathcal{C})$  is commutative, the  $T_l(U)$  commute, so we may add the exponents in the product of exponentials. The unified exponent is

$$\begin{aligned}
& \sum_{U' \in I(\mathcal{C})} \left\langle [U'], \log \left( \left( 1 + \sum_U p_l(\underline{x}^{(U)})[U] \right) \left( 1 + \sum_V p_l(\underline{y}^{(V)})[V] \right) \right) \right\rangle \otimes T_l(U') \\
&= \sum_{U' \in I(\mathcal{C})} \left\langle [U'], \log \left( 1 + \sum_U p_l \left( \underline{x}^{(U)}, \underline{y}^{(U)}, \bigoplus_{V_1, V_2 \in I(\mathcal{C})} (\underline{x}^{(V_1)} \underline{y}^{(V_2)})^{\oplus N_{V_1, V_2}^U} \right) [U] \right) \right\rangle \otimes T_l(U')
\end{aligned}$$

Here we have used direct sum notation to indicate that  $p_l$  should have a collection of symmetric function variables as arguments, and the direct sum in the exponents denotes the multiplicity of each of the sets of variables. For convenience we write  $\underline{z}^{(U)}$  to denote the family of symmetric function variables  $\bigoplus_{V_1, V_2 \in I(\mathcal{C})} (\underline{x}^{(V_1)} \underline{y}^{(V_2)})^{\oplus N_{V_1, V_2}^U}$ . Note that if the variables  $\underline{x}$  are indexed as  $x_i$ , we have

$$E(t) = \sum_{r \geq 0} e_r(\underline{x}) t^r = \prod_i (1 + x_i t)$$

So  $E(t)$  (and in particular  $E(-1)$ ) is multiplicative with respect to variable sets.

$$\left( \sum_{r \geq 0} (-1)^r e_r(\underline{x}^{(1)}) \right) \left( \sum_{s \geq 0} (-1)^s e_s(\underline{y}^{(1)}) \right) = \left( \sum_{r \geq 0} (-1)^r e_r(\underline{x}^{(1)}, \underline{y}^{(1)}) \right)$$

Thus our original generating function becomes

$$\left( \sum_{r \geq 0} (-1)^r e_r(\underline{x}^{(1)}, \underline{y}^{(1)}) \right) \prod_{l=1}^{\infty} \exp \left( \sum_{U' \in I(\mathcal{C})} \left\langle [U'], \log \left( 1 + \sum_U p_l(\underline{x}^{(U)}, \underline{y}^{(U)}, \underline{z}^{(U)})[U] \right) \right\rangle \otimes T_l(U') \right)$$

This is very close to the generating function of Theorem 9.10 in variables  $\underline{x}^{(U)}, \underline{y}^{(U)}, \underline{z}^{(U)}$  (only the leading factor is different). We write it as

$$\frac{1}{\sum_{r \geq 0} (-1)^r e_r(\underline{z}^{(1)})} \sum_{\lambda \in \mathcal{P}_c} \left( \prod_{U \in I(\mathcal{C})} s_{\lambda(U)}(\underline{x}^{(U)}, \underline{y}^{(U)}, \underline{z}^{(U)}) \right) \otimes X_{\lambda}$$

If the variables  $\underline{x}^{(U)}$  and  $\underline{y}^{(V)}$  are indexed as  $x_i^{(U)}$  and  $y_j^{(V)}$  respectively, the leading term can also be written

$$\prod_{U, V \in I(\mathcal{C})} \left( \prod_{i, j} \frac{1}{1 - x_i^{(U)} y_j^{(V)}} \right)^{N_{U, V}^{(1)}} = \prod_{U, V \in I(\mathcal{C})} \left( \sum_{\rho \in \mathcal{P}} s_{\rho}(\underline{x}^{(U)}) s_{\rho}(\underline{y}^{(V)}) \right)^{N_{U, V}^{(1)}}$$

This completes the proof.  $\square$

10.1. **The Case of  $\mathcal{C} = \mathbf{Vect}(k)$ .** Now we specialise to the case where  $\mathcal{C}$  is the category of finite-dimensional vector spaces over  $k$ . In that case there is only one  $U \in I(\mathcal{C})$ , namely  $k$  which is idempotent with respect to the tensor structure (which we omit from the notation for convenience). Also,  $\mathcal{P}_{\mathcal{C}}$  can be identified with  $\mathcal{P}$ . To illustrate how to perform this computation in Theorem 10.2, we use it to prove the following theorem of Littlewood [Lit58].

**Theorem 10.3.** *The reduced Kronecker coefficients satisfy the following identity:*

$$\tilde{k}_{\mu,\nu}^{\lambda} = \sum_{\sigma_1, \sigma_2, \sigma_3} \sum_{\rho_1, \rho_2, \rho_3} k_{\sigma_2, \sigma_3}^{\sigma_1} c_{\sigma_1, \rho_2, \rho_3}^{\lambda} c_{\rho_1, \sigma_2, \rho_3}^{\mu} c_{\rho_1, \rho_2, \sigma_3}^{\nu}$$

Here  $k_{\rho_1, \rho_2}^{\rho_3}$  is a Kronecker coefficient, and  $c_{\alpha, \beta, \gamma}^{\delta}$  is a (generalised) Littlewood-Richardson coefficient (it is the coefficient of  $s_{\delta}$  in  $s_{\alpha} s_{\beta} s_{\gamma}$ ). All sums are over the set of all partitions.

*Proof.* We consider the case where  $\mathcal{C}$  is the category of finite-dimensional vector spaces over  $k$ . In that case there is only one  $U \in I(\mathcal{C})$ , namely  $k$  which is idempotent with respect to the tensor structure (which we omit from the notation for convenience). Also,  $\mathcal{P}_{\mathcal{C}}$  can be identified with  $\mathcal{P}$ . Thus  $\underline{z}^{(k)}$  is just  $\underline{xy}$ . Below, all sums are over the set of partitions.

$$\begin{aligned} & \left( \sum_{\rho_1} s_{\rho_1}(\underline{x}) s_{\rho_1}(\underline{y}) \right) \sum_{\lambda} s_{\lambda}(\underline{x}, \underline{y}, \underline{xy}) \otimes X_{\lambda} \\ &= \sum_{\rho_1} s_{\rho_1}(\underline{x}) s_{\rho_1}(\underline{y}) \sum_{\lambda, \rho_2, \rho_3} \sum_{\sigma_1} c_{\sigma_1 \rho_2, \rho_3}^{\lambda} (s_{\rho_3}(\underline{x}) s_{\rho_2}(\underline{y}) s_{\sigma_1}(\underline{xy})) \otimes X_{\lambda} \\ &= \sum_{\rho_1, \rho_2, \rho_3} s_{\rho_1}(\underline{x}) s_{\rho_1}(\underline{y}) \sum_{\lambda} \sum_{\sigma_1, \sigma_2, \sigma_3} c_{\sigma_1 \rho_2, \rho_3}^{\lambda} k_{\sigma_2, \sigma_3}^{\sigma_1} (s_{\rho_3}(\underline{x}) s_{\rho_2}(\underline{y}) s_{\sigma_3}(\underline{x}) s_{\sigma_2}(\underline{y})) \otimes X_{\lambda} \\ &= \sum_{\rho_1, \rho_2, \rho_3} \sum_{\sigma_1, \sigma_2, \sigma_3} k_{\sigma_2, \sigma_3}^{\sigma_1} \sum_{\lambda} c_{\sigma_1, \rho_2, \rho_3}^{\lambda} \sum_{\mu} c_{\rho_1, \sigma_2, \rho_3}^{\mu} s_{\mu}(\underline{x}) \sum_{\nu} c_{\rho_1, \rho_2, \sigma_3}^{\nu} s_{\nu}(\underline{y}) \otimes X_{\lambda} \end{aligned}$$

This completes the proof.  $\square$

We also point out that Theorem 9.10 gives a generating function for a known family of symmetric functions. As above we omit  $U = k$  entirely from our notation, as well as the tensor product symbols. Theorem 9.10 becomes

**Theorem 10.4.** *We have the following equality of generating functions.*

$$\sum_{\lambda \in \mathcal{P}} s_{\lambda} X_{\lambda} = \left( \sum_{i \geq 0} (-1)^i e_i \right) \prod_{l \geq 1} (1 + p_l)^{T_l}$$

Let the variables of the symmetric functions present in the above expression be  $\underline{x}$ . We introduce a new set of symmetric functions in the variables  $\underline{y}$  such that the  $i$ -th basic hook is identified with  $e_i(\underline{y})$ . We write  $\tilde{s}_{\lambda}(\underline{y})$  for the image of  $X_{\lambda}$  in this correspondence. In accordance with Proposition 9.9 we have the following equality.

$$\begin{aligned} \sum_{\lambda \in \mathcal{P}} s_{\lambda}(\underline{x}) \tilde{s}_{\lambda}(\underline{y}) &= \left( \sum_{i \geq 0} (-1)^i e_i(\underline{x}) \right) \prod_{l \geq 1} (1 + p_l(\underline{x}))^{\frac{1}{l} \sum_{d|l} \mu(l/d) p_d(\underline{y})} \\ &= \left( \sum_{i \geq 0} (-1)^i e_i(\underline{x}) \right) \prod_{l \geq 1} \sum_{r \geq 0} p_l(\underline{x})^r \binom{\frac{1}{l} \sum_{d|l} \mu(l/d) p_d(\underline{y})}{r} \end{aligned}$$

The  $\tilde{s}_{\lambda}$  are a polynomial in the elementary symmetric functions such that if the  $i$ -th elementary symmetric function is replaced with the  $i$ -th exterior power of the permutation representation of  $S_n$ , and the multiplication is taken to be the tensor product of  $S_n$ -representations, then for  $n$  sufficiently large, the virtual representation we obtain is the Specht module  $S^{\tilde{\lambda}(n)}$  (this also implies that the characters are obtained by evaluating these symmetric functions at suitable roots of unity). Thus the  $\tilde{s}_{\lambda}$  are fundamental objects in



the asymptotic representation theory of symmetric groups. A combinatorial description of them is given in [OZ16]; comparing the above generating function with their Proposition 11, combined with the description of character polynomials in Example 14 of Section 7 of [Mac95] makes it clear that these are indeed the same symmetric functions.

## 11. GENERALISATION TO TENSOR CATEGORIES

All our results thus far are valid in the setting where  $\mathcal{C}$  is a ring category, as per Definition 4.2.3 on page 66 of [EGNO15], and in particular for any tensor category. That is,  $\mathcal{C}$  is an essentially small, locally finite  $k$ -linear abelian monoidal category satisfying two conditions. Firstly, if  $\mathbf{1}$  is the unit object, then  $\text{End}_{\mathcal{C}}(\mathbf{1}) = k$ . Secondly, the product in  $\mathcal{C}$  is exact in both arguments and bilinear with respect to direct sums. The essentially small property allows the construction of the Grothendieck group  $\mathcal{G}(\mathcal{C})$ , whilst the artinian property implies that the  $\mathcal{G}(\mathcal{C})$  is the free abelian group generated by isomorphism classes of simple objects. The exactness of the product in the category implies that it respects the relations of the Grothendieck group and therefore descends to a bilinear distributive multiplication on  $\mathcal{G}(\mathcal{C})$ . Thus,  $\mathcal{G}(\mathcal{C})$ , inherits the structure of a ring. Due to a theorem of Takeuchi, an essentially small  $k$ -linear Artinian abelian category (in particular, our  $\mathcal{C}$ ) is equivalent to  $C\text{-comod}$  for some coalgebra  $C$  over  $k$  [Tak77].

The category of finite-dimensional modules for a bialgebra over  $k$  is an example of a ring category (as is the category of finite-dimensional comodules). Generalising this, the category of finite-dimensional modules over a quasibialgebra is also a ring category.

In order to construct wreath product categories, we make use of Deligne's tensor product for categories, which we briefly describe. If  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are  $k$ -linear artinian categories, then their tensor product,  $\mathcal{C}_1 \boxtimes \mathcal{C}_2$  is another artinian category. It is equipped with a bifunctor  $\boxtimes : \mathcal{C}_1 \times \mathcal{C}_2 \rightarrow \mathcal{C}_1 \boxtimes \mathcal{C}_2$  satisfying a certain universal property; details can be found in [EGNO15]. For our purposes, it suffices to know several properties. Firstly, simple objects in  $\mathcal{C}_1 \boxtimes \mathcal{C}_2$  are precisely those of the form  $S_1 \boxtimes S_2$  where  $S_1$  and  $S_2$  are simple objects of  $\mathcal{C}_1$  and  $\mathcal{C}_2$ , respectively. This is a consequence of the fact that if  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are coalgebras such that  $\mathcal{C}_1$  is equivalent to  $C_1\text{-comod}$  and  $\mathcal{C}_2$  is equivalent to  $C_2\text{-comod}$ , then  $\mathcal{C}_1 \boxtimes \mathcal{C}_2$  is equivalent to  $(C_1 \otimes C_2)\text{-comod}$ . Secondly, if  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are tensor categories, then so is  $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ , with tensor structure arising from  $(X_1 \boxtimes Y_1) \otimes (X_2 \boxtimes Y_2) = (X_1 \otimes X_2) \boxtimes (Y_1 \otimes Y_2)$ .

**Example 11.1.** *If  $A_1$  and  $A_2$  are finite-dimensional  $k$ -algebras, then  $(A_1\text{-mod}) \boxtimes (A_2\text{-mod}) = (A_1 \otimes_k A_2)\text{-mod}$ .*

We may form the  $n$ -fold Deligne's tensor product of  $\mathcal{C}$  which is itself a ring category, which we denote  $\mathcal{C}^{\boxtimes n}$ .

**Definition 11.2.** *The equivariantisation of  $\mathcal{C}^{\boxtimes n}$  under the natural action of  $S_n$  is the wreath product category  $\mathcal{W}_n(\mathcal{C}) = (\mathcal{C}^{\boxtimes n})^{S_n}$ . If  $\mathcal{C}$  is a ring category, then  $\mathcal{W}_n(\mathcal{C})$  obtains the structure of a ring category.*

**Example 11.3.** *If  $A$  is a finite-dimensional  $k$ -algebra then  $\mathcal{W}_n(A\text{-mod})$  is equivalent to  $(A \wr S_n)\text{-mod}$ , the category of finite-dimensional modules for the wreath product (although  $A$  would need some additional structure for  $A\text{-mod}$  to be a ring category).*

In our situation, it is important to consider actions of subgroups of  $S_n$ . If a group  $G$  acts on objects of  $\mathcal{C}$ , so does any subgroup, via restriction. Following [Mor12], if  $\mathcal{D}$  is an additive category, then for any subgroup  $H$  of finite index in  $G$ , we have a forgetful functor  $\text{Res}_H^G : \mathcal{D}^G \rightarrow \mathcal{D}^H$ . Additionally there is an induction functor  $\text{Ind}_H^G : \mathcal{D}^H \rightarrow \mathcal{D}^G$  which is both right adjoint and left adjoint to  $\text{Res}_H^G$ . The induction functor may be written as a sum over coset representatives of  $H$  in  $G$  as follows:

$$\text{Ind}_H^G(M) = \bigoplus_{g \in G/H} gM$$

In the above formula, the action of  $G$  is analogous to that of an induced representation of a finite group. The main case of interest is when  $G = S_n$  and  $H$  is a Young subgroup of  $S_n$ .

An identical classification of simple objects (i.e. specific objects induced from Young subgroups in the

sense described above) of  $\mathcal{W}_n(\mathcal{C})$  holds in this greater generality. In [Mor12], this is shown in the context of indecomposable objects of an additive category, but the proof in this setting is analogous.

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